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Beskonačne formule za π bez riječi

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Leibnizova formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \ldots$$

Dokaz:

$$\frac{\pi}{4} = \operatorname{arctg} 1 = \int_0^1 \frac{dx}{1+x^2} = \int_0^1 (1 - x^2 + x^4 - x^6 + \ldots)\, dx =$$

$$= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)\Bigg|_0^1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Eulerova formula

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}}$$

Dokaz:

$$b_n = a_1 + a_1a_2 + a_1a_2a_3 + \ldots + a_1a_2a_3\ldots a_n \Rightarrow$$

$$\Rightarrow b_n = a_1\,(1 + a_2\,(1 + a_3\,(\ldots\,(1 + a_n)\ldots)))$$

$$\frac{b_n}{b_n + 1} = \frac{a_1(1 + a_2(1 + a_3(\ldots(1 + a_n)\ldots)))}{a_1(1 + a_2(1 + a_3(\ldots(1 + a_n)\ldots))) + 1} =$$

$$= \frac{a_1}{a_1 + \cfrac{1}{a_2(1 + a_3(\ldots(1 + a_n)\ldots)) + 1}} =$$

$$= \frac{a_1}{a_1 + 1 - \cfrac{a_2(1 + a_3(\ldots(1 + a_n)\ldots))}{a_2(1 + a_3(\ldots(1 + a_n)\ldots)) + 1}} =$$

$$= \cfrac{a_1}{a_1 + 1 - \cfrac{a_2}{a_2 + 1 - \cfrac{a_3}{\ddots \, \cfrac{}{a_{n-1} + 1 - \cfrac{a_n}{a_{n+1}}}}}} =$$

$$a_1 = -\frac{1}{3}, \quad a_2 = -\frac{3}{5}, \quad a_3 = -\frac{5}{7}, \; \ldots \; a_n = -\frac{2n-1}{2n+1} \Rightarrow$$

$$\Rightarrow b_n = -\frac{1}{3} + \frac{1}{5} - \ldots \pm \frac{1}{2n+1}$$

$$\frac{b_n}{b_n + 1} = \cfrac{-1/3}{-1/3 + 1 + \cfrac{3/5}{-3/5 + 1 + \cfrac{5/7}{\ddots \, \cfrac{}{-\frac{2n-3}{2n-1} + 1 - \cfrac{(2n-1)/(2n+1)}{(2n-1)/(2n+1) - 1}}}}} =$$

$$= \cfrac{-1}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{\ddots \, \cfrac{}{2 + \cfrac{(2n-1)^2}{2}}}}} \Rightarrow$$

$$\Rightarrow \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{\ddots \, \cfrac{}{2 + \cfrac{(2n-1)^2}{2}}}}}} =$$

$$= \frac{1}{1-\frac{b_n}{b_n+1}} = 1 + b_n = 1 - \frac{1}{3} + \frac{1}{5} - \ldots \pm \frac{1}{2n+1} \to \frac{\pi}{4}$$

Wallisova formula

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\,\frac{4\cdot 4}{3\cdot 5}\,\frac{6\cdot 6}{5\cdot 7}\,\frac{8\cdot 8}{7\cdot 9}\,\frac{10\cdot 10}{9\cdot 11}\cdots$$

Dokaz:

$$\underbrace{\int_0^{\pi/2} \sin^n x\,dx}_{I_n} = \underbrace{-\frac{1}{n}\sin^{n-1}x\cos x\Big|_0^{\pi/2}}_{0} + \underbrace{\frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx}_{I_{n-2}}$$

$$0 < x < \pi/2 \;\Rightarrow\; 0 < \sin x < 1 \;\Rightarrow\; I_{n+1} < I_n$$

$$\left.\begin{aligned} I_0 &= \frac{\pi}{2} \;\Rightarrow\; I_{2n} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\ldots\cdot\frac{1}{2}\cdot\frac{\pi}{2} \\ I_1 &= 1 \;\Rightarrow\; I_{2n+1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\ldots\cdot\frac{2}{3}\cdot 1 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow\; I_{2n+1} < I_{2n} < I_{2n-1} \;\Rightarrow\; 1 < \frac{I_{2n}}{I_{2n+1}} < \frac{I_{2n-1}}{I_{2n+1}} = \frac{2n+1}{2n} = 1 + \frac{1}{2n} \;\Rightarrow$$

$$\Rightarrow\; \frac{I_{2n+1}}{I_{2n}} = \frac{2n\cdot 2n}{(2n+1)(2n-1)}\,\frac{(2n-2)(2n-2)}{(2n-1)(2n-3)}\cdots\frac{2\cdot 2}{3\cdot 1}\,\frac{2}{\pi} \to 1 \;\Rightarrow$$

$$\Rightarrow\; \frac{\pi}{2} = \frac{2\cdot 2}{3\cdot 1}\,\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2n\cdot 2n}{(2n-1)(2n+1)}\cdots$$

Vièteova formula

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

Dokaz:

$$\begin{aligned} \sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} = 2^2\sin\frac{x}{2^2}\cos\frac{x}{2^2}\cos\frac{x}{2} = \\ &= 2^3\sin\frac{x}{2^3}\cos\frac{x}{2^3}\cos\frac{x}{2^2}\cos\frac{x}{2} = \cdots = \\ &= 2^n\sin\frac{x}{2^n}\cos\frac{x}{2^n}\cos\frac{x}{2^{n-1}}\cdots\cos\frac{x}{2^2}\cos\frac{x}{2} \end{aligned}$$

$$\frac{\sin x}{2^n\sin(x/2^n)} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}$$

$$x = \frac{\pi}{2} \quad \Rightarrow \quad \frac{1}{2^n \sin(\pi/2^{n+1})} = \cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cdots\cos\frac{\pi}{2^{n+1}}$$

$$\left.\begin{array}{l} \dfrac{1}{2^n \sin(\pi/2^{n+1})} = \dfrac{2}{\pi}\,\dfrac{\pi/2^{n+1}}{\sin(\pi/2^{n+1})} \quad \to \quad \dfrac{2}{\pi} \\[2ex] \cos\dfrac{\alpha}{2} = \sqrt{\dfrac{1}{2} + \dfrac{1}{2}\cos\alpha} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad \frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdot \ldots$$