A NOTE ON THE GEOMETRY OF $\int_a^b x^a\,dx = \left.\frac{x^{a+1}}{a+1}\right|_a^b$

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As every student of mathematics knows, the geometrical meaning of the formula in the title (for $\alpha \neq -1$) is that the area from a to b, under x^α, equals the difference of the two areas

$$\int_*^b x^a\,dx = \frac{b^{a+1}}{a+1} \qquad \text{and} \qquad \int_*^a x^a\,dx = \frac{a^{a+1}}{a+1}$$

where $*$ is the appropriate reference point. The reference point is not the same for every α (which is not so well known among the students):

$$* = 0, \quad \text{for } \alpha > -1 \qquad\qquad * = \infty, \quad \text{for } \alpha < -1.$$

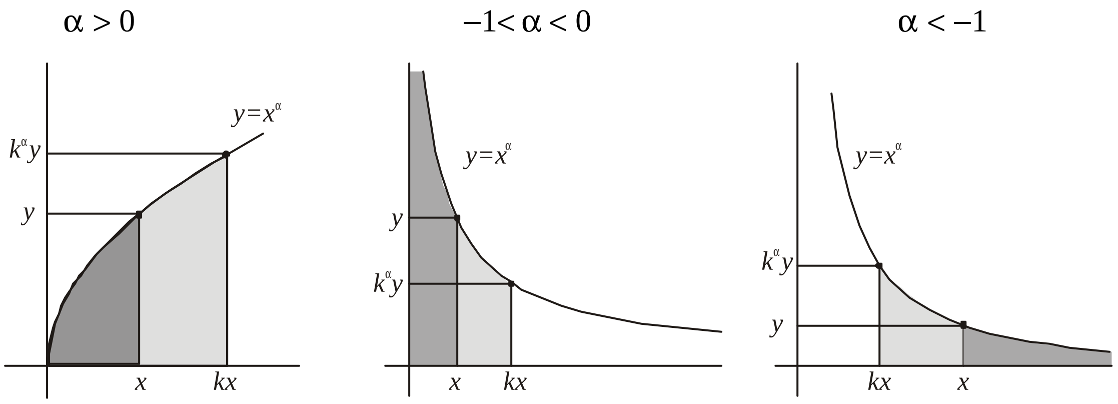
Using the geometrical idea of stretching it is quite easy to prove that the area from $*$ to x, under x^α, has the following form:

$$\text{(1)} \qquad \int_*^x x^a\,dx = x^{a+1}p(a)\,; \qquad p(a) = \int_*^1 x^a\,dx\,.$$

Namely, if we stretch (x, y)-plane by factor k in x-direction and by factor k^α in y-direction, then it is easy to prove that every area A will be stretched by factor $k^{\alpha+1}$, i.e.

$$(x, y) \to (kx, k^\alpha y) \quad \Rightarrow \quad A \to k^{\alpha+1}A$$

If we apply this result to the areas under x^α (the black area is stretched into the gray), we get our formula:



$$\int_0^{kx} x^a\,dx = k^{a+1}\int_0^{x} x^a\,dx \qquad \int_\infty^{kx} x^a\,dx = k^{a+1}\int_\infty^{x} x^a\,dx$$

Substituing $* = 0$, for $\alpha > -1$, and $* = \infty$, for $\alpha < -1$, we have:

$$\int_*^{kx} x^a\,dx = k^{a+1}\int_*^{x} x^a\,dx\,,$$

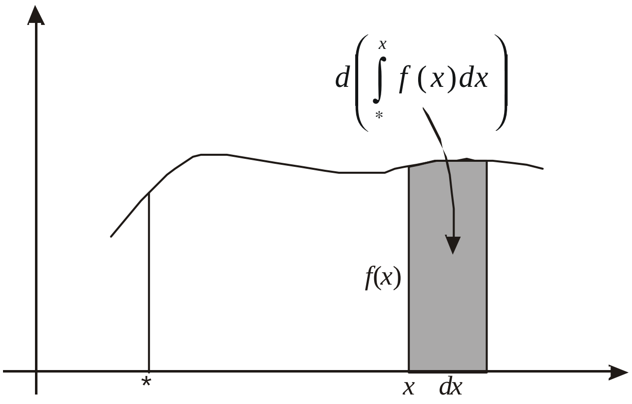
which (for $x = 1$) reduces to:

$$\int_*^{k} x^a\,dx = k^{a+1}\int_*^{1} x^a\,dx\,.$$

This is a variant of our formula (1), with the variable (*stretching*) factor $x^{\alpha+1}$ and constant (*unit*) factor $p(\alpha)$.

From the integral formula (1), according to the fundamental theorem of calculus, which also has the clear geometrical meaning (cf. the figure below):

$$\frac{d\left(\int_*^{x} f(x)dx\right)}{dx} = f(x)$$



there follows the differential formula $x^{\alpha} = p(\alpha)\, dx^{\alpha+1}/dx$, i. e.

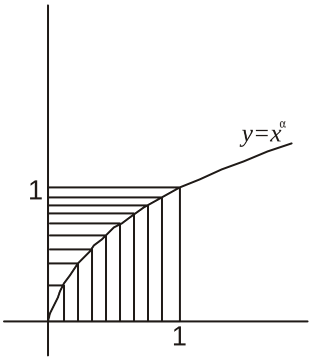
$$(2) \qquad \frac{dx^{a}}{dx} = \frac{x^{a-1}}{p(a-1)}; \qquad p(a) = \int_{*}^{1} x^{a}\, dx .$$

There is lot of ways to prove that $p(\alpha) = 1/(1+\alpha)$, for $\alpha \neq -1$. We introduce two geometrical proves, which we find quite elegant.

I.

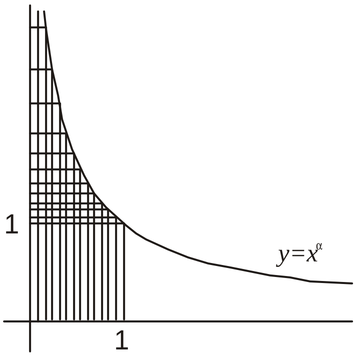
If $y = x^{\alpha}$ then $x = y^{1/\alpha}$ and it follows that:

$\alpha > 0$



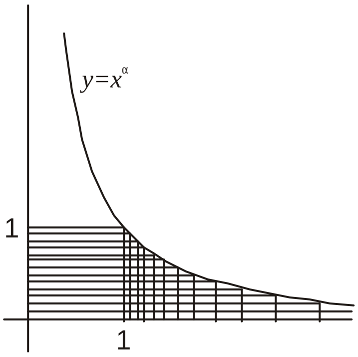
$$\int_{0}^{1} x^{a}\, dx + \int_{0}^{1} y^{1/a}\, dy = 1$$

$-1 < \alpha < 0$



$$\int_{0}^{1} x^{a}\, dx + \int_{\infty}^{1} y^{1/a}\, dy = 1$$

$\alpha < -1$



$$\int_{\infty}^{1} x^{a}\, dx + \int_{0}^{1} y^{1/a}\, dy = 1$$

Substituing $* = 0$, and $* = \infty$, in appropriate places, we have:

$$\int_{*}^{1} x^{a}\, dx + \int_{*}^{1} y^{1/a}\, dy = 1, \ \ \alpha \neq -1.$$

Hence, if we define $p(1/0) = p(\infty) = 0$, we have:

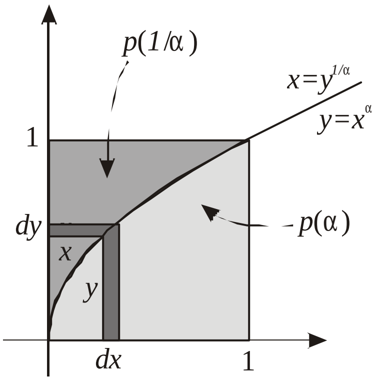
$$(3) \qquad p(\alpha) + p(1/\alpha) = 1, \ \ \alpha \neq -1.$$

On the other hand, from $y = x^{\alpha}$ and (2) it follows:

$$\frac{dy}{dx} = \frac{x^{a-1}}{p(a-1)} = \frac{y/x}{p(a-1)}, \qquad \text{i. e. } \frac{x\,dy}{y\,dx} = \frac{1}{p(a-1)}.$$

Taking into account that $\frac{x\,dy}{y\,dx}$ is constant, it follows (cf. the figure below) that

$$\frac{x\,dy}{y\,dx} = \frac{p(1/a)}{p(a)}$$



Hence, we have

(4) $$\frac{p(1/a)}{p(a)} = \frac{1}{p(a-1)}.$$

From (3) and (4) follows

$$\frac{1}{p(a)} - 1 = \frac{1}{p(a-1)}.$$

If we substitute L(α) for the left hand side, we get:

$$\mathrm{L}(\alpha) = \mathrm{L}(\alpha - 1) + 1.$$

But $p(0) = 1$, i. e. $\mathrm{L}(0) = 0$, from which it follows that $\mathrm{L}(n) = n$ i.e.

(5) $$p(n) = \frac{1}{1+n}, \; n \in \mathrm{N}.$$

If $\alpha = n/m \in \mathrm{Q}^+$ then $y = x^\alpha = x^{n/m}$, which means that $y^m = x^n$. From (2) and (5), it follows that,

$$my^{m-1}dy = nx^{n-1}dx, \quad \text{i. e.} \quad \frac{x\,dy}{y\,dx} = \frac{n}{m}.$$

Taking into account (cf. above) that

$$\frac{x\,dy}{y\,dx} = \frac{p(1/a)}{p(a)} \ ,$$

we have

$$\frac{p(1/a)}{p(a)} = \frac{n}{m}. \tag{6}$$

From (3) and (6),

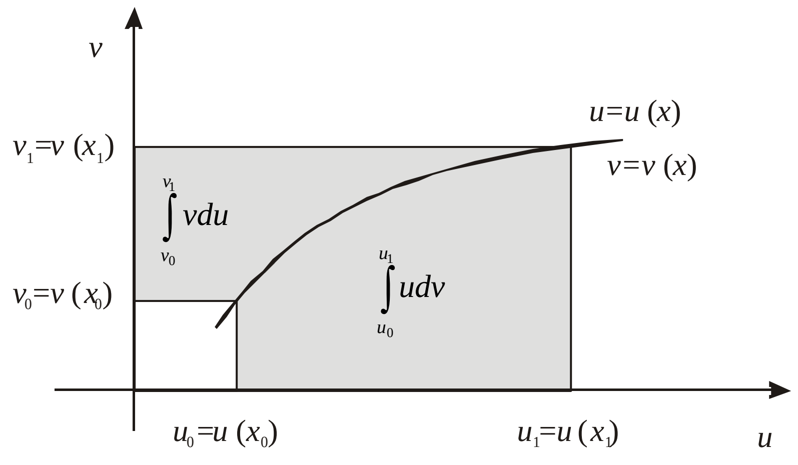
$$\frac{1}{p(a)} - 1 = \frac{n}{m}, \qquad p(a) = \frac{1}{1+\dfrac{n}{m}}.$$

Hence, $p(\alpha) = 1/(1+\alpha)$ for every $\alpha \in Q^{+}$. In a similar way $p(\alpha) = 1/(1+\alpha)$ can be proved for $\alpha \in Q^{-}$ and this could be done as an exercise by the reader.

II.

In our second and more elegant proof of $p(\alpha) = 1/(1+\alpha)$, we use partial integration; which also has a clear geometrical meaning (cf. the figure below):

$$\int_{x_0}^{x_1} u(x)dv(x) + \int_{x_0}^{x_1} v(x)du(x) = u(x)v(x)\Big|_{x_0}^{x_1}$$



Our second proof is as follows:

$$p(a+b)=\int_*^1 x^{a+b}\,dx=\int_*^1 x^a x^b\,dx=\begin{pmatrix} u=x^a & dv=x^b\,dx \\ du\overset{(2)}{=}\dfrac{x^{a-1}}{p(a-1)}dx & v\overset{(1)}{=}p(b)x^{b+1}\end{pmatrix}=$$

$$p(b)\,x^{a+b+1}\Big|_*^1-\frac{p(b)}{p(a-1)}\int_*^1 x^{a+b}\,dx= p(b)-\frac{p(b)}{p(a-1)}\,p(a+b)\,.$$

Hence,

(7) $$p(a+b)=\frac{p(b)}{1+\dfrac{p(b)}{p(a-1)}}\,.$$

Substituting β = 0 and taking into account that $p(0) = 1$, we get:

(8) $$\frac{1}{p(a-1)}=\frac{1}{p(a)}-1$$

Substituting (8) in (7) we get:

$$p(a+b)=\frac{p(b)}{1+p(b)((1/p(a))-1)}=\frac{p(a)p(b)}{p(a)+p(b)-p(a)p(b)}\,.$$

It follows,

$$\frac{1}{p(a+b)}=\frac{1}{p(a)}+\frac{1}{p(b)}-1\,.$$

If we substitute L(α) for $(1/p(\alpha)) - 1$, we get:

(9) $$\mathrm{L}(\alpha+\beta) = \mathrm{L}(\alpha) + \mathrm{L}(\beta).$$

The only continuous solution of this functional equation is $\mathrm{L}(\alpha) = k\alpha$. From $p(1) = 1/2$ it follows that L(1) = 1 i. e. $k = 1$.

Hence,

$$L(\alpha) = \frac{1}{p(a)} - 1 = \alpha, \qquad \text{i. e.} \qquad p(\alpha) = \frac{1}{1+a}\,.$$

The proof is valid for every $\alpha \neq -1$, if we presuppose the continuity of $p(\alpha)$. Without that assumption the proof is valid for every $\alpha \in Q\backslash\{-1\}$, because (9) implies linearity of L on Q.