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CONTINUING VARIATIONS ON A SYSTEM OF GENTZEN

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Gentzen has shown in [1] that, in his sequent version of predicate logic, the distinction between IPC and CPC is formally represented as distinction between singular and multiple version of his system. Kleene has shown in [2] that singularity of the rules $(\to\neg)$ and $(\to T)$ will suffice for IPC. López-Escobar has shown in [3] that the singularity of $(\to\neg)$ alone, or $(\to T)$ alone will not suffice for IPC, because putting any of them in multiple form yields CPC. He has also shown (in [3] again) that any combination of singular and multiple rules will yield IPC, CD or CPC.

The method of Kleene and López-Escobar is to start from multiple version of G1 (that is from CPC) and to find the restricted versions (restricted by the singularity conditions imposed on some of the rules) adequate for IPC (Kleene) and CD (López-Escobar). With such an attitude López-Escobar calls any singularity restriction an intuitionistic restriction (in [3]), and using this terminology we may summarize the above mentioned results:

1) intuitionistic restrictions on $(\to T)$ and $(\to\neg)$ are adequate for IPC,

2) intuitionistic restrictions on $(\to\neg)$ and $(\to\supset)$ (but not on $(\to T)$) are adequate for CD (but not for IPC),

3) intuitionistic restrictions on $(\to T)$ alone or on $(\to\neg)$ alone are not adequate for IPC.

But with such an attitude we can not answer the following question: Which of the rules of G1, in multiple form, are intuitionistically permissible, and which of them are strictly classical (that means, which of them will convert an otherwise intuitionistic system into a classical one)?

Take for example $(\to T)$ in multiple form. Is it intuitionistically permissible? Kleene's result 1) or López-Escobar's 2) will suggest that this rule is not intuitionistically permissible. But the assertion that the intuitionistic restriction on $(\to\supset)$, $(\to\neg)$ and $(\to\forall)$ are adequate for IPC[1]) will suggest that the rule $(\to T)$ is intuitionistically permissible. Must the answer to our question be the relative one? No, we assert that there is the absolute answer to our question; but to get it we have to start, contrary to Kleene and López-Escobar, from singular version of G1 (we call it G_s1) in which every rule of G1 is in singular form (that is, we have to start from IPC). The system G_s1 is the metatheory of Gentzen's NI (which is one of the most natural versions

[1]) I could not locate the (first) appearance of this result in the literature but it has been proved, in effect, in Theorem 1 below. (Our proof, contrary to the above mentioned method of restricting the CPC, proceeds by conservatively extending IPC. This paper should make clear the advantage of such a proof.)

of IPC) and it is known (from [1]) that

$$\Gamma \vdash_{\mathrm{NI}} C \quad \text{iff} \quad \vdash_{\mathrm{G_s1}} \Gamma \to C.$$[1])

There is no multiple conclusion in NI, hence there is no multiple sequent in G_s1. But if we want to make sense of multiple rules we (first) have to make sense of multiple sequents (sequents with more than one formula in succedent). We fix the meaming of such sequents by the following rules:

$(\mathrm{I_m})$ $\quad \dfrac{\Gamma \to \Theta^{\vee}}{\Gamma \to \Theta}$ introduction of multiple sequent,

$(\mathrm{E_m})$ $\quad \dfrac{\Gamma \to \Theta}{\Gamma \to \Theta^{\vee}}$ elimination of multiple sequent,

where $\Theta^{\vee}$ is disjunction of formulae in Θ, because the intended meaning (classical and intuitionistic) of more than one formula in succedent is disjunctive.

Remark. The use of the construct $(\ )^{\vee}$ in the formulation of $(\mathrm{I_m})$ and $(\mathrm{E_m})$ is not essential. We could do with

(I'_m) $\quad \dfrac{\Gamma \to A \vee B, \Theta}{\Gamma \to A, B, \Theta}$

(E'_m) $\quad \dfrac{\Gamma \to A, B, \Theta}{\Gamma \to A \vee B, \Theta}$

but we find it easier to work with $(\mathrm{I_m})$ and $(\mathrm{E_m})$ instead. Note also that when using the construct $(\ )^{\vee}$ we presuppose the associativity of $\vee$, which is provable in G_s1. (In the following we will use the associativity of $\vee$ without mention.)

The system G_s1 extended with the rules $(\mathrm{I_m})$ and $(\mathrm{E_m})$ will be called $G_{sm}1$. This system is a conservative extension of G_s1 as the following lemma asserts.

Lemma 1. *For each singular sequent* $\Gamma \to C$

$$\vdash_{\mathrm{G_{sm}1}} \Gamma \to C \quad \textit{iff} \quad \vdash_{\mathrm{G_s1}} \Gamma \to C.$$

Moreover, for each sequent $\Gamma \to \Theta$

$$\vdash_{\mathrm{G_{sm}1}} \Gamma \to \Theta \quad \textit{iff} \quad \vdash_{\mathrm{G_s1}} \Gamma \to \Theta^{\vee}.$$

Proof. Straightforward verification.

If follows that $G_{sm}1$ is an intuitionistic system in which multiple sequents aquire sense and this system therefore provides a proper framework for our question. Namely, a multiple rule of G1 is intuitionistically permissible iff it is permissible as a rule in $G_{sm}1$.

Now, it is easy to show that all the rules of G1 (in multiple form) except $(\to\supset)$, $(\to\neg)$ and $(\to\forall)$ are permissible in $G_{sm}1$ as the following theorem asserts.

Theorem 1. *Each application of the rules of* G1 *(in multiple form) excluding the rules* $(\to\supset)$, $(\to\neg)$ *and* $(\to\forall)$ *is realizable by applications of the rules of* $G_{sm}1$.

[1]) Gentzen made no use of the fact that G_s1 is the metatheory of NI, although this fact would make the proof of the assertion very transparent (trivial indeed).

Proof.

$(\supset\to)$

$$\dfrac{\dfrac{\Delta \to \Lambda, A}{\Delta \to \Lambda^{\vee} \vee A} \quad \dfrac{\dfrac{\dfrac{\Lambda^{\vee} \to \Lambda^{\vee}}{\Lambda^{\vee} \to \Lambda^{\vee} \vee \Theta^{\vee}}}{\overline{\Lambda^{\vee}, A \supset B, \Gamma \to \Lambda^{\vee} \vee \Theta^{\vee}}} \quad \dfrac{\dfrac{\dfrac{A \to A B \to B}{A \supset B, A \to B} \quad \dfrac{\dfrac{B, \Gamma \to \Theta}{B, \Gamma \to \Theta^{\vee}}}{B, \Gamma \to \Lambda^{\vee} \vee \Theta^{\vee}}}{A \supset B, A, \Gamma \to \Lambda^{\vee} \vee \Theta^{\vee}}}{A, A, \supset B, \Gamma \to \Lambda^{\vee} \vee \Theta^{\vee}}}{\Lambda^{\vee} \vee A, A \supset B, \Gamma \to \Lambda^{\vee} \vee \Theta^{\vee}}}{\dfrac{\dfrac{\Delta, A \supset B, \Gamma \to \Lambda^{\vee} \vee \Theta^{\vee}}{\overline{A \supset B, \Delta, \Gamma \to \Lambda^{\vee} \vee \Theta^{\vee}}}}{A \supset B, \Delta, \Gamma \to \Lambda, \Theta}}$$

$(\to\&)$

$$\dfrac{\dfrac{\Gamma \to \Theta, B}{\Gamma \to \Theta^{\vee} \vee B} \quad \dfrac{\dfrac{\dfrac{\dfrac{\Theta^{\vee} \to \Theta^{\vee}}{A, \Theta^{\vee} \to \Theta^{\vee}}}{\Theta^{\vee}, A \to \Theta^{\vee}}}{\Theta^{\vee}, A \to \Theta^{\vee} \vee (A \,\&\, B)} \quad \dfrac{\dfrac{\dfrac{A \to A}{B, A \to A} \quad \dfrac{\dfrac{B \to B}{A, B \to B}}{B, A \to B}}{B, A \to A \,\&\, B}}{B, A \to \Theta^{\vee} \vee (A \,\&\, B)}}{\Theta^{\vee} \vee B, A \to \Theta^{\vee} \vee (A \,\&\, B)}}{\dfrac{\Gamma, A \to \Theta^{\vee} \vee (A \,\&\, B)}{\vdots}}$$

$$\dfrac{\dfrac{\Gamma \to \Theta, A}{\Gamma \to \Theta^{\vee} \vee A} \quad \dfrac{\Theta^{\vee}, \Gamma \to \Theta^{\vee} \vee (A \,\&\, B) \quad \dfrac{\vdots}{A, \Gamma \to \Theta^{\vee} \vee (A \,\&\, B)}}{\Theta^{\vee} \vee A, \Gamma \to \Theta^{\vee} \vee (A \,\&\, B)}}{\dfrac{\dfrac{\Gamma, \Gamma \to \Theta^{\vee} \vee (A \,\&\, B)}{\overline{\Gamma \to \Theta^{\vee} \vee (A \,\&\, B)}}}{\Gamma \to \Theta, A \,\&\, B}}$$

$(\&\to)$

$$\dfrac{\dfrac{\dfrac{A, \Gamma \to \Theta}{A, \Gamma \to \Theta^{\vee}}}{A \,\&\, B, \Gamma \to \Theta^{\vee}}}{A \,\&\, B, \Gamma \to \Theta} \qquad \dfrac{\dfrac{\dfrac{B, \Gamma \to \Theta}{B, \Gamma \to \Theta^{\vee}}}{A \,\&\, B, \Gamma \to \Theta^{\vee}}}{A \,\&\, B, \Gamma \to \Theta}$$

$(\to\vee)$

$$\dfrac{\dfrac{\dfrac{\Gamma \to \Theta, A}{\Gamma \to \Theta^{\vee} \vee A}}{\Gamma \to \Theta^{\vee} \vee A \vee B}}{\Gamma \to \Theta, A \vee B}$$

$$\dfrac{\dfrac{\dfrac{\Gamma\to\Theta,B}{\Gamma\to\Theta^\vee\vee B}}{\Gamma\to\Theta^\vee\vee B\vee A}\qquad \dfrac{\dfrac{\Theta^\vee\to\Theta^\vee}{\Theta^\vee\to\Theta^\vee\vee A\vee B}\quad \dfrac{\dfrac{\dfrac{B\to B}{B\to A\vee B}\quad\dfrac{A\to A}{A\to A\vee B}}{B\vee A\to A\vee B}}{B\vee A\to\Theta^\vee\vee A\vee B}}{\Theta^\vee\vee B\vee A\to\Theta^\vee\vee A\vee B}}{\dfrac{\Gamma\to\Theta^\vee\vee A\vee B}{\Gamma\to\Theta,A\vee B}}$$

$(\vee\to)$

$$\dfrac{\dfrac{\dfrac{A,\Gamma\to\Theta}{A,\Gamma\to\Theta^\vee}\qquad\dfrac{B,\Gamma\to\Theta}{B,\Gamma\to\Theta^\vee}}{A\vee B,\Gamma\to\Theta^\vee}}{A\vee B,\Gamma\to\Theta}$$

$(\neg\to)$

$$\dfrac{\dfrac{\Gamma\to\Theta,A}{\Gamma\to\Theta^\vee\vee A}\qquad\dfrac{\dfrac{\dfrac{\Theta^\vee\to\Theta^\vee}{\neg A,\Theta^\vee\to\Theta^\vee}}{\Theta^\vee,\neg A\to\Theta^\vee}\quad\dfrac{\dfrac{\dfrac{A\to A}{\neg A,A\to}}{A,\neg A\to}}{A,\neg A\to\Theta^\vee}}{\Theta^\vee\vee A,\neg A\to\Theta^\vee}}{\dfrac{\dfrac{\Gamma,\neg A\to\Theta^\vee}{\overline{\neg A,\Gamma\to\Theta^\vee}}}{A,\Gamma\to\Theta}}$$

$(\forall\to)$

$$\dfrac{\dfrac{\dfrac{A(t),\Gamma\to\Theta}{A(t),\Gamma\to\Theta^\vee}}{\forall xA(x),\Gamma\to\Theta^\vee}}{\forall xA(x),\Gamma\to\Theta}$$

$(\to\exists)$

$$\dfrac{\dfrac{\Gamma\to\Theta,A(t)}{\Gamma\to\Theta^\vee\vee A(t)}\qquad\dfrac{\dfrac{\Theta^\vee\to\Theta^\vee}{\Theta^\vee\to\Theta^\vee\vee\exists xA(x)}\quad\dfrac{\dfrac{A(t)\to A(t)}{A(t)\to\exists xA(x)}}{A(t)\to\Theta^\vee\vee\exists xA(x)}}{\Theta^\vee\vee A(t)\to\Theta^\vee\vee\exists xA(x)}}{\dfrac{\Gamma\to\Theta^\vee\vee\exists xA(x)}{\Gamma\to\Theta,\exists xA(x)}}$$

$(\exists\to)$

$$\dfrac{\dfrac{\dfrac{A(b),\Gamma\to\Theta}{A(b),\Gamma\to\Theta^\vee}}{\exists xA(x),\Gamma\to\Theta^\vee}\,(*)}{\exists xA(x),\Gamma\to\Theta}\quad .$$

(→T)
$$\dfrac{\dfrac{\dfrac{\Gamma\to\Theta}{\Gamma\to\Theta^{\vee}}}{\Gamma\to\Theta^{\vee}\vee C}}{\Gamma\to\Theta, C}$$

(T→)
$$\dfrac{\dfrac{\dfrac{\Gamma\to\Theta}{\Gamma\to\Theta^{\vee}}}{C,\Gamma\to\Theta^{\vee}}}{C,\Gamma\to\Theta}$$

(→C)
$$\dfrac{\dfrac{\dfrac{\Gamma\to\Theta, C, C}{\Gamma\to\Theta^{\vee}\vee C\vee C}\qquad\dfrac{\Theta^{\vee}\vee C\to\Theta^{\vee}\vee C\qquad\dfrac{C\to C}{C\to\Theta^{\vee}\vee C}}{\Theta^{\vee}\vee C\vee C\to\Theta^{\vee}\vee C}}{\Gamma\to\Theta^{\vee}\vee C}}{\Gamma\to\Theta, C}$$

(C→)
$$\dfrac{\dfrac{\dfrac{C, C,\Gamma\to\Theta}{C, C,\Gamma\to\Theta^{\vee}}}{C,\Gamma\to\Theta^{\vee}}}{C,\Gamma\to\Theta}$$

(→I)
$$\dfrac{\dfrac{\dfrac{C\to C}{C\to C\vee\Theta^{\vee}}}{C\to D\vee C\vee\Theta^{\vee}}\qquad\dfrac{\dfrac{D\to D}{D\to D\vee C\vee\Theta^{\vee}}\qquad\dfrac{\Theta^{\vee}\to\Theta^{\vee}}{\Theta^{\vee}\to D\vee C\vee\Theta^{\vee}}}{D\vee\Theta^{\vee}\to D\vee C\vee\Theta^{\vee}}}{C\vee D\vee\Theta^{\vee}\to D\vee C\vee\Theta^{\vee}}$$
$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\Gamma\to\Lambda, C, D,\Theta}{\Gamma\to\Lambda^{\vee}\vee C\vee D\vee\Theta^{\vee}}\qquad\dfrac{\dfrac{\Lambda^{\vee}\to\Lambda^{\vee}}{\Lambda^{\vee}\to\Lambda^{\vee}\vee D\vee C\vee\Theta^{\vee}}\qquad\begin{matrix}\vdots\\ C\vee D\vee\Theta^{\vee}\to\Lambda^{\vee}\vee D\vee C\vee\Theta^{\vee}\end{matrix}}{\Lambda^{\vee}\vee C\vee D\vee\Theta^{\vee}\to\Lambda^{\vee}\vee D\vee C\vee\Theta^{\vee}}}{\Gamma\to\Lambda^{\vee}\vee D\vee C\vee\Theta^{\vee}}}{\Gamma\to\Lambda, D, C,\Theta}$$

(I→)
$$\dfrac{\dfrac{\dfrac{\Delta, D, C,\Gamma\to\Theta}{\Delta, D, C,\Gamma\to\Theta^{\vee}}}{\Delta, C, D,\Gamma\to\Theta^{\vee}}}{\Delta, C, D,\Gamma\to\Theta}$$

(Cut)
$$\dfrac{\dfrac{\dfrac{\Delta\to\Lambda, C}{\Delta\to\Lambda^{\vee}\vee C}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\Lambda^{\vee}\to\Lambda^{\vee}}{\Lambda^{\vee}\to\Lambda^{\vee}\vee\Theta^{\vee}}}{\Gamma,\Lambda^{\vee}\to\Lambda^{\vee}\vee\Theta^{\vee}}}{\Lambda^{\vee},\Gamma\to\Lambda^{\vee}\vee\Theta^{\vee}}\qquad\dfrac{\dfrac{C,\Gamma\to\Theta}{C,\Gamma\to\Theta^{\vee}}}{C,\Gamma\to\Lambda^{\vee}\vee\Theta^{\vee}}}{\Lambda^{\vee}\vee C,\Gamma\to\Lambda^{\vee}\vee\Theta^{\vee}}}{\Delta,\Gamma\to\Lambda^{\vee}\vee\Theta^{\vee}}}{\Delta,\Gamma\to\Lambda,\Theta}$$

Remark. Note that the multiple form of $(\to \mathrm{T})$ is intuitionistically permissible (quite trivialy, as a matter of fact).

The following question remains: Which of the rules $(\to\supset)$, $(\to\neg)$ and $(\to\forall)$ (in multiple form) is strictly classical?

We get CPC from IPC by adding *tertium non datur* and therefore the proper framework for this question is the system $G_{sm}1$ extended with the axiom scheme

$$\to A \vee \neg A .$$

It is easy to show that each application of the rules $(\to\supset)$, $(\to\neg)$ and $(\to\forall)$ (in multiple form) is realizable in the extended system:

$(\to\supset)$

$$\dfrac{\to A \vee \neg A \qquad \dfrac{\dfrac{A, \Gamma \to \Theta, B \qquad \dfrac{\dfrac{B \to B}{A, B \to B}}{B \to A \supset B}}{A, \Gamma \to \Theta, A \supset B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A, B}}{\neg A, A \to B}}{A, \neg A \to B}}{\neg A \to A \supset B}}{\neg A \to A \supset B, \Theta}}{\overline{\neg A \to \Theta, A \supset B}}}{\overline{\Gamma, \neg A \to \Theta, A \supset B}}}{\overline{\neg A, \Gamma \to \Theta, A \supset B}}}{A \vee \neg A, \Gamma \to \Theta, A \supset B}}{\Gamma \to \Theta, A \supset B}$$

$(\to\neg)$

$$\dfrac{\to A \vee \neg A \qquad \dfrac{\dfrac{A, \Gamma \to \Theta}{A, \Gamma \to \Theta, \neg A} \qquad \dfrac{\dfrac{\neg A \to \neg A}{\overline{\Gamma, \neg A \to \neg A, \Theta}}}{\overline{\neg A, \Gamma \to \Theta, \neg A}}}{A \vee \neg A, \Gamma \to \Theta, \neg A}}{\Gamma \to \Theta, \neg A}$$

$(\to\forall)$

$$\dfrac{\dfrac{\dfrac{\to \Theta^{\vee} \vee \neg \Theta^{\vee}}{\to \Theta^{\vee}, \neg \Theta^{\vee}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to \Theta, A(b)}{\Gamma \to \Theta^{\vee} \vee A(b)}}{\Gamma \to \Theta^{\vee}, A(b)}}{\Gamma \to A(b), \Theta^{\vee}}}{\neg \Theta^{\vee}, \Gamma \to A(b)}}{\neg \Theta^{\vee}, \Gamma \to \forall x A(x)}\ (*)}{\Gamma \to \Theta^{\vee}, \forall x A(x)}}{\overline{\Gamma \to \Theta, \forall x A(x)}}$$

(Here we have freely used the multiple forms of the rules, the applications of which are realizable in $G_{sm}1$ by Theorem 1.)

The following theorem, which explains what happens in the other direction, gives answer to our question.

Theorem 2. 1) *It is possible to prove each sequent of the form* $\to A \vee \neg A$ *in* $G_{sm}1$ *extended with* $(\to\neg)$ *in multiple form. This rule is, therefore, strictly classical.* 2) *It is possible to prove each sequent of the form* $\to A \vee \neg A$ *in* $G_{sm}1$ *extended with* $(\to\supset)$ *in multiple form. This rule is, therefore, strictly classical.* 3) *It is not possible to prove each sequent of the form* $\to A \vee \neg A$ *in* $G_{sm}1$ *extended with* $(\to\forall)$ *in multiple form. Therefore, this rule is not strictly classical (it suffices for CD only; cf.* [3], Lemma 2.1[1])).

Proof.

1)
$$\dfrac{\dfrac{A \to A}{\to A, \neg A}}{\to A \vee \neg A}$$

2)
$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A, A \,\&\, \neg A}}{\to A, A \supset (A \,\&\, \neg A)} \qquad \dfrac{\dfrac{\dfrac{A \to A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}}{A \,\&\, \neg A, A \to}}{A, A \,\&\, \neg A \to}}{A \,\&\, \neg A, A \,\&\, \neg A \to}}{A \,\&\, \neg A \to}}{A \supset (A \,\&\, \neg A), A \to}}{A, A \supset (A \,\&\, \neg A) \to}}{A \supset (A \,\&\, \neg A) \to \neg A}}{\to A, \neg A}}{\to A \vee \neg A}$$

(Here, again, we have freely used the multiple forms of the rules, the applications of which are realizable in $G_{sm}1$ by Theorem 1.)

3) The proof of the unprovability of $\to \mathscr{A} \vee \neg \mathscr{A}$ (for an atomic letter $\mathscr{A}$) in intuitionistic G3 ([2], p. 483) can be repeated in intuitionistic G3 extended with multiple form of $(\to\forall)$ without any difficulties. Now, by straightforward verification, we see that the unprovability of $\to \mathscr{A} \vee \neg \mathscr{A}$ in G3 plus $(\to\forall)$ (in multiple form) implies the unprovability of $\to \mathscr{A} \vee \neg \mathscr{A}$ in $G_{sm}1$ plus $(\to\forall)$ (in multiple form).

One final question. How does it happen that $\to$T aquire a special status with KLEENE and LÓPEZ-ESCOBAR in spite of the fact that it is on a par with other intuitionistically permissible multiple rules (as shown by our presentation of the problem of intuitionistic permissibility of multiple rules)? The answer is quite simple. KLEENE knew (from [1]) that multiple version of G1 is classical, and that the singular version G_s1 is intuitionistic. But he also realizes (in [2], p. 444, Lemma 32a) that (in multiple G1) starting from singular axioms we can reach a multiple sequent only by applying the rule $(\to\neg)$ or $(\to$T$)$. So, if we restrict these rules to be singular we can never reach a multiple sequent and, therefore, this restriction yields an intuitionistic system. But

[1]) There is a misprint in the formulation of Lemma 2.1. Instead of $\vdash \forall x(A(x) \vee B) \to \forall x(A(x) \vee B)$ there should be $\vdash \forall x(A(x) \vee B) \to \forall x A(x) \vee B$, as it is in the proof.

by imposing the singularity restriction only on the rules $(\rightarrow\neg)$ and $(\rightarrow\mathrm{T})$ we are, so to say, only pretending that we are admitting the other rules to be multiple, because this only restriction does not admit a multiple application of any of the rules. As KLEENE remarks (in [2], p. 445), it is useful to observe that we need to impose the singularity restriction only on the two of the rules, so as to impose it in effect on all of them. But if we are interested in intuitionistic permissibility of multiple rules, we have to make sense of multiple sequents (and in this way of multiple rules) and only then we can explore the question of permissibility of these rules (in the way we did). We saw that this exploration leads to the conclusion that the singularity restrictions which deserve the name of intuitionistic restrictions are (contrary to LÓPEZ-ESCOBAR's usage of the term) only the singularity restrictions on $(\rightarrow\supset)$, $(\rightarrow\neg)$ and $(\rightarrow\forall)$.

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