

EULER'S FORMULA IS ALMOST UNIQUE

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Any extension of a number system should be performed in accordance with Hankel's principle¹⁾:

Hankel's principle

A definition of an operation should be extended from a restricted domain to a wider one in such a way as to conserve the crucial algebraic properties of the operation.

The crucial algebraic properties of addition and multiplication are the well known properties of commutativity, associativity and distributivity and their conservation compels us to the standard definitions of addition and multiplication in Z , Q and C . In the same way the crucial algebraic properties of exponentiation:

$$(1) \quad a^{b+c} = a^b \cdot a^c \quad (a^b)^c = a^{b \cdot c} \quad (a \cdot b)^c = a^c \cdot b^c$$

compels us to define the exponentiation in Z and Q in the usual way. For example,

$$2^{-3} = \frac{1}{2^3} \quad \text{because} \quad 2^3 \cdot 2^{-3} = 2^{3-3} = 2^0 = 1,$$

$$2^{\frac{3}{2}} = \sqrt{2^3} \quad \text{because} \quad \left(2^{\frac{3}{2}}\right)^2 = 2^{\left(\frac{3}{2}\right) \cdot 2} = 2^3.$$

We want to prove that the crucial algebraic properties (1) do not compel us to define exponentiation in C in accordance with Euler's formula

$$(2) \quad e^{i\omega} = \cos \omega + i \sin \omega,$$

but that it **almost** compels us. Namely, it compels us to the formula

$$(3) \quad e^{i\omega} = \cos(c\omega) + i \sin(c\omega),$$

in which c can be any real constant.

Complex numbers are obtained by adding the imaginary unit i to the reals and by combining the old reals with the new unit i , using the operations $+$ and \cdot uniquely extended in accordance with Hankel's principle. Hence, it is immediately realized that any element of the new complex domain is of the form $x + iy$ for real x and y (because of the defining property of $i : i^2 = -1$) and that the totality of all new numbers forms a field. Our first question is whether it is possible to define exponentiation of complex numbers in accordance with Hankel's principle, so as to remain within the complex domain?²⁾ We prove it is.

First notice that $-i$ has the same defining property as $i : (-i)^2 = -1$. So, any calculation with i , which ends with the result $R(i) = x + iy$, if performed on $-i$ will end with the result $R(-i) = x - iy$. Exponentiation is a calculation process in the complex domain, hence, for real a and ω if $R(-i) = a^{i\omega} = x + iy$ then $R(-i) = a^{-i\omega} = x - iy$. But then, by Hankel's principle, we have

$$a^{i\omega} \cdot a^{-i\omega} = a^{i\omega-i\omega} = a^0 = 1 = (x + iy) \cdot (x - iy) = x^2 + y^2$$

which means that

$$a^{i\omega} = \cos \phi + i \sin \phi.$$

It remains to find out how ϕ depends on a and ω . First we prove that

$$(4) \quad \phi(a, k \cdot \omega) = k \cdot \phi(a, \omega).$$

From Hankel's principle and DeMorgan's formula it follows:

$$\begin{aligned} \cos \phi(a, k\omega) + i \sin \phi(a, k\omega) &= a^{ik\omega} = (a^{i\omega})^k = (\cos \phi(a, \omega) + i \sin \phi(a, \omega))^k = \\ &= (\cos k\phi(a, \omega) + i \sin k\phi(a, \omega)) \end{aligned}$$

Next we prove that

$$(5) \quad \phi(a^k, \omega) = k \cdot \phi(a, \omega).$$

Again, from Hankel's principle and DeMorgan's formula it follows:

$$\begin{aligned} \cos \phi(a^k, \omega) + i \sin \phi(a^k, \omega) &= (a^k)^{i\omega} = (a^{i\omega})^k = (\cos \phi(a, \omega) + i \sin \phi(a, \omega))^k = \\ &= \cos k\phi(a, \omega) + i \sin k\phi(a, \omega) \end{aligned}$$

If continuity of exponentiation is to be preserved in the complex domain $\phi(a, \omega)$ has to be continuous in a and ω . Hence, the continuity will be presupposed in the sequel. But then it follows from (4) that $\phi(a, \omega)$ is linear in the second argument:

$$(6) \quad \phi(a, \omega) = k(a) \cdot \omega.$$

It follows from (5) that $\phi(a, \omega)$ is linear in the logarithm of the first argument:

$$(7) \quad \phi(a, \omega) = \ln a \cdot h(\omega).$$

From (6) and (7) we have

$$k(a) \cdot \omega = \ln a \cdot h(\omega)$$

which means that for any real a and ω

$$\frac{k(a)}{\ln a} = \frac{h(\omega)}{\omega} \quad \text{i.e.} \quad \frac{k(a)}{\ln a} = \frac{h(\omega)}{\omega} = c = \text{const.}$$

Hence, it follows from (6) or (7) that

$$\phi(a, \omega) = c \cdot \omega \cdot \ln a.$$

We conclude that the only possible definition of exponentiation in the complex domain, which is in accordance with Hankel's principle, is the following one

$$a^{i\omega} = \cos(c\omega \ln a) + i \sin(c\omega \ln a).$$

It is also easy to see that the crucial algebraic properties (1) are really preserved by this definition for any choice of c .

In particular, we are compelled by Hankel's principle to define

$$e^{i\omega} = \cos(c\omega) + i \sin(c\omega).$$

Hence, Euler's formula is not the unique formula which is in accordance with Hankel's principle but we could say it is almost unique (up to the constant c which we can choose arbitrarily).

Could it be that we are compelled to choose $c = 1$ if we want to define exponentiation of complex base with complex exponent in accordance with Hankel's principle? No, we are not. Let $z_1 = r(\cos \varphi + i \sin \varphi)$ and let $z_2 = x + iy$. Then, by Hankel's principle and DeMorgan's formula, we have:

$$\begin{aligned} z_1^{z_2} &= (r(\cos \varphi + i \sin \varphi))^{(x+iy)} = r^{(x+iy)}(\cos \varphi + i \sin \varphi)^{(x+iy)} \\ &= r^x(\cos(cy \ln r) + i \sin(cy \ln r))(\cos(x\varphi) + i \sin(x\varphi))(\cos \varphi + i \sin \varphi)^{iy} \\ &= r^x(\cos(x\varphi + cy \ln r) + i \sin(x\varphi + cy \ln r))(e^{i\varphi/c})^{iy} \\ &= r^x e^{-y\varphi/c}(\cos(x\varphi + cy \ln r) + i \sin(x\varphi + cy \ln r)), \end{aligned}$$

and it is easy to see that the crucial algebraic properties (1) are preserved by the definition:

$$(r (\cos \varphi + i \sin \varphi))^{(x+iy)} = r^x e^{-y\varphi/c} (\cos(x\varphi + cy \ln r) + i \sin(x\varphi + cy \ln r))$$

for any choice of c . Hence, Hankel's principle does not compel us to choose Euler's $c = 1$.

Finally, we prove that we are compelled to choose $c = 1$ if we extend Hankel's principle to the differentiability. Namely, it is only for $c = 1$ that the function $f(z) = a^z$ is differentiable. It is easy to prove this.

The function

$$u + iv = a^{x+iy} = a^x \cos(cy \ln a) + i a^x \sin(cy \ln a)$$

is differentiable only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which means, only if

$$a^x \ln a \cos(cy \ln a) = c a^x \ln a \cos(cy \ln a)$$

which means, only if

$$c = 1.$$

We may conclude. **By Hankel's principle we are almost compelled to Euler's formula. We are definitely compelled to it if we extend the principle to the**

differentiability. Notice, that we need only differentiability, which is a relaxation of the usual conditions used in more common derivations of Euler's formula e.g. $de^{i\omega}/d\omega = ie^{i\omega}$ in [2], or $d\bar{e}^z/dz = \bar{e}^z$ in [3].

- 1) It was announced by G. Peacock in [1], six years before Hankel was born. As everybody knows, mathematical attributions are usually wrong.
- 2) Notice, that this is not possible for rational numbers. If we define $2^{1/2}$ in accordance with Hankel's principle as $\sqrt{2}$ we do not remain within rationals.

[1] Peacock G.: *Report on the Recent Progress and Present State of Certain Branches of Analysis*, Third Meeting of the British Association for the Advancement of Science, 1833.

[2] Allendoerfer, C. B.: *The Proof of Euler's Equation*, American Mathematical Monthly, vol.55, 1948.

[3] Curtiss, D. R.: *Analytic Functions of a Complex Variable*, MAA, 1926.