THE IRONY OF THE STANDARD PROOF OF \( \frac{d\sin \varphi}{d\varphi} = \cos \varphi \)

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The arclength of the part of the unit circle is defined using polygons with small sides:

\[
\varphi = \lim_{\Delta s \to 0} (\Sigma \Delta s)
\]

From this definition it follows that

\[
\lim_{\Delta \varphi \to 0} \frac{\Delta s}{\Delta \varphi} = 1,
\]

If \( x = \cos \varphi \) and \( y = \sin \varphi \) are defined as coordinates of the point on the unit circle with the corresponding arclength \( \varphi \), then for their derivatives we have:

\[
\frac{d}{d\varphi} \sin \varphi = \frac{dy}{d\varphi} = \lim_{\Delta \varphi \to 0} \frac{\Delta y}{\Delta \varphi} = \lim_{\Delta \varphi \to 0} \frac{\Delta y}{\Delta s} \frac{\Delta s}{\Delta \varphi} = \lim_{\Delta \varphi \to 0} \frac{\Delta y}{\Delta s} \lim_{\Delta \varphi \to 0} \frac{\Delta s}{\Delta \varphi} = \lim_{\Delta \varphi \to 0} \frac{\Delta y}{\Delta s} = \lim_{\Delta \varphi \to 0} (\cos \alpha) = \cos \varphi.
\]

The companion formula \( \frac{d\cos \varphi}{d\varphi} = -\sin \varphi \) can be proved in the same way.
How did we manage to prove \( \frac{d\sin \varphi}{d\varphi} = \cos \varphi \), in such a simple way? Where did the infamous

\[
(2) \quad \lim_{\Delta \varphi \to 0} \frac{\sin \Delta \varphi}{\Delta \varphi} = 1
\]

disappeared? Of course, it is contained in the definition of the arclength \( \varphi \), i.e. it is contained in (1). (Formulae (1) and (2) are obviously equivalent.) Basically, the standard proof of \( \frac{d\sin \varphi}{d\varphi} = \cos \varphi \) is difficult because it take pains in establishing (2) although it already presupposes (2) in the form (1). Namely, it presupposes it in the very definition of \( \varphi \) which is, of course, presupposed in the definition of \( \sin \varphi \). Really ironic.