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MULTIPLE FORMS OF GENTZEN'S RULES AND SOME INTERMEDIATE LOGICS

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Gentzen's sequential system is a formalization of classical or intuitionistic logic depending on whether we take its rules in multiple or singular form. Indeed, in the singular system extended by the initial sequents of the form $\to A \vee \neg A$, it is possible to prove at once the permissibility of the multiple forms of all the inference rules [1].

An analysis of each rule separately shows that the multiple form of the introduction of negation or implication in the succedent is sufficient for the formalization of classical logic. The multiple form of the introduction of the universal quantifier in the succedent is not sufficient for the formalization of classical logic, and at the same time it is too strong for the formalization of intuitionistic logic [2, p. 487, Theorem 58]. The multiple forms of the other rules do not extend the intuitionistic system. The extension of the singular (intuitionistic) system by the multiple form of the introduction of the universal quantifier in the succedent is therefore the formalization of an intermediate logic. We call this extension L_2.

We want to show that the system L_2 is related to Gödel's completeness theorem. In fact, Kleene's detailed analysis of the proof of the theorem [3, pp. 283–312], reveals that the only non-intuitionistic assertion used in the proof is of the form $\forall x A(x) \vee \exists x \neg A(x)$, for the predicate $A(x)$ which informally has the following meaning: "After x rounds in the construction of the sequent tree, the sequent tree is not terminated" (we have used the terminology of [3, pp. 295–305]). Therefore, we will compare our system L_2 to the singular (intuitionistic) system extended by the initial sequents of the form $\to \forall x A(x) \vee \exists x \neg A(x)$. We call this extension L_3. Moreover, Kleene's analysis shows that the predicate $A(x)$ is decidable. Therefore, we will also compare our system L_2 to the singular (intuitionistic) system extended by the initial sequents of the form $\forall x(A(x) \vee \neg A(x)) \to \forall x A(x) \vee \exists x \neg A(x)$. We call this extension L_1.

We prove constructively the following theorem:

Theorem. L_3 *extends* L_2, *and* L_2 *extends* L_1. (*It is plain that* L_3 *properly extends* L_1 [2, p. 487].)

Before we prove the theorem we have to clarify some notions: What does it mean to say that the multiple system LK (of [1]) is equivalent to the system LI (of [1]) extended by the initial sequents of the form $\to A \vee \neg A$ (further on, this system will be referred to as LI+)? We cannot simply say that every sequent provable in LK is provable in LI+, and vice versa, because it is trivially true that in the singular system LI+ it is impossible to prove even a single sequent with more than one formula in the succedent. But, LI+ is equivalent to LK in the following sense:

A sequent $\Gamma \to A_1, \ldots, A_n$ is provable in LK iff $\Gamma \to A_1 \vee \ldots \vee A_n$ is provable in LI+.

To prove this, we have to notice that: 1) the multiple system LK with rules of the form $\dfrac{\Gamma \to A_1, \ldots, A_n}{\Delta \to B_1, \ldots, B_k}$ and initial sequents of the form $\Gamma \to A_1, \ldots, A_n$ is trivially equivalent (in the formerly explained way) to a singular system LKS which has its corresponding rules of the form $\dfrac{\Gamma \to A_1 \vee \ldots \vee A_n}{\Delta \to B_1 \vee \ldots \vee B_k}$ and its corresponding initial sequents of the form $\Gamma \to A_1 \vee \ldots \vee A_n$, and that: 2) all the rules and initial sequents of LKS are permissible in LI+.

In this connection, all the assertions of the first two paragraphs have to be interpreted in this way. For example:

1) "The multiple form of the introduction of negation in the succedent $\dfrac{\Gamma, A \to \Theta}{\Gamma \to \neg A, \Theta}$ is sufficient for the formalization of classical logic" means that LKS is equivalent to LI+ extended by the rule $\dfrac{\Gamma, A \to B}{\Gamma \to \neg A \vee B}$.

2) "The multiple form of the introduction of universal quantifier in the succedent $\dfrac{\Gamma \to A(a), \Theta}{\Gamma \to \forall x A(x), \Theta}$ (a does not occur in the conclusion) is not sufficient for the formalization of classical logic, and at the same time it is too strong for the formalization of intuitionistic logic" means that LKS has more provable sequents than LI extended by the rule $\dfrac{\Gamma \to \forall x A(x) \vee B}{\Gamma \to A(a) \vee B}$ (a does not occur in the conclusion), which on the other hand has more provable sequents than LI alone.

In the following proof we have to bear in mind these clarifications.

Proof of the theorem.

(I) We prove that L_2 extends L_1.

Note that L_2 is LI extended by the rule

$$\frac{\Gamma \to A(a) \vee B}{\Gamma \to \forall x A(x) \vee B} \text{ (a does not occur in the conclusion)}$$

while L_1 is LI extended by the initial sequents of the form

$$\forall x(A(x) \vee \neg A(x)) \to \forall x A(x) \vee \exists x \neg A(x).$$

Thus we can prove our first claim if we find a scheme for proofs in L_2 of sequents of the form

$$\forall x(A(x) \vee \neg A(x)) \to \forall x A(x) \vee \exists x \neg A(x).$$

Here is the scheme:

$$
\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a)}{A(a) \to A(a) \vee \exists x \neg A(x)} \qquad \dfrac{\dfrac{\neg A(a) \to \neg A(a)}{\neg A(a) \to \exists x \neg A(x)}}{\neg A(a) \to A(a) \vee \exists x \neg A(x)}}{A(a) \vee \neg A(a) \to A(a) \vee \exists x \neg A(x)}}{\vdots}
\qquad
\dfrac{\forall x(A(x) \vee \neg A(x)) \to \forall x(A(x) \vee \neg A(x))}{\forall x(A(x) \vee \neg A(x)) \to A(a) \vee \neg A(a)} \quad \vdots}
{\dfrac{\forall x(A(x) \vee \neg A(x)) \to A(a) \vee \exists x \neg A(x)}{\forall x(A(x) \vee \neg A(x)) \to \forall x A(x) \vee \exists x \neg A(x)}\ (1)}
$$

(1) Here we have used the multiple form of the introduction of the universal quantifier in the succedent.

(II) We prove that L_3 extends L_2.

Note that L_3 is LI extended by initial sequents of the form $\to \forall x A(x) \vee \exists x \neg A(x)$. Thus, to prove our second claim we will prove that the rule

$$
\frac{\Gamma \to A(a) \vee B}{\Gamma \to \forall x A(x) \vee B} \qquad (a \text{ does not occur in the conclusion})
$$

is permissible in L_3. Here we show how to pass from $\Gamma \to A(a) \vee B$ to $\Gamma \to \forall x A(x) \vee B$ (a does not occur in $\Gamma \to \forall x A(x) \vee B$) in L_3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x(A(x) \vee B) \to \forall x(A(x) \vee B)}{\forall x(A(x) \vee B) \to A(a) \vee B}}{\neg A(a), \forall x(A(x) \vee B) \to B}\ (1)}{\exists x \neg A(x), \forall x(A(x) \vee B) \to B}\ (2)}{\exists x \neg A(x), \forall x(A(x) \vee B) \to \forall x A(x) \vee B}
$$

$$
\vdots
$$

$$
\dfrac{\to \forall x A(x) \vee \exists x \neg A(x) \qquad (3)\ \dfrac{\dfrac{\dfrac{\forall x A(x) \to \forall x A(x)}{\forall x A(x) \to \forall x A(x) \vee B}}{\forall x A(x), \forall x(A(x) \vee B) \to \forall x A(x) \vee B} \quad \vdots}{\forall x A(x) \vee \exists x \neg A(x), \forall x(A(x) \vee B) \to \forall x A(x) \vee B}}{\forall x(A(x) \vee B) \to \forall x A(x) \vee B}
$$

$$
\vdots
$$

$$
\dfrac{\dfrac{\Gamma \to A(a) \vee B}{\Gamma \to \forall x(A(x) \vee B)}\ (4) \qquad \vdots}{\Gamma \to \forall x A(x) \vee B}
$$

(1) This is the multiple form of the introduction of negation in the antecedent. It is permissible in LI, hence in L_3.

(2) The variable a does not occur in $\Gamma \to \forall x A(x) \vee B$; hence it does not occur in $\exists x \neg A(x), \forall x(A(x) \vee B) \to B$. Thus the restriction on variables is not violated by this step.

(3) $\to \forall x A(x) \vee \exists x \neg A(x)$ is an initial sequent of L_3.

(4) This is the singular form of the introduction of the universal quantifier in the succedent.

This completes the proof.

The question remains: Is the system L_2 equivalent to L_1 or possibly to L_3?[1])

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[1]) Added in the proof: K. Došen proved that L_3 is equivalent to LK. Thanks to his remark we also realized that in part (I) of the proof of our theorem, (1) represents an instance of the rule

$$\mathrm{d}(\to\forall)\ \frac{\Gamma \to A(a) \vee B}{\forall x(A(x) \vee \neg A(x)), \Gamma \to \forall x A(x) \vee B}.$$

Hence we proved there that LI + $\mathrm{d}(\to\forall)$ extends L_1. Replacing the scheme (3) by scheme $\forall x(A(x) \vee \neg A(x)) \to \forall x A(x) \vee \exists x \neg A(x)$ in part (II) of the proof, we prove (thereby) that L_1 extends LI+$\mathrm{d}(\to\forall)$. Hence L_1 = LI+$\mathrm{d}(\to\forall)$. The singular $(\to\forall)$ rule is dispensible in LI+$\mathrm{d}(\to\forall)$.

References

[1] Gentzen, G., Untersuchungen über das logische Schließen. Math. Zeitschr. **39** (1935), 176–210, 405–431.

[2] Kleene, S. C., Introduction to Metamathematics. D. van Nostrand Comp., New York 1952.

[3] Kleene, S. C., Mathematical Logic. J. Wiley, New York 1968.

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