VII. Mathematikertreffen Zagreb–Graz
Grazer Math. Ber. 313 (1991)

ON DEVIDÉ'S AXIOMATIZATION OF NATURAL NUMBERS

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Devidé proposed a very simple axiomatization of natural numbers in [De]. He characterizes the successor function S, and it's domain N, with the following axioms:

(S0) $(\exists!m)m\in(N\setminus S(N))$

(D) $M = S(M)\longrightarrow M = \emptyset$.

Fraenkel refered to this axiomatization as "... a weaker formulation..." (cf. [FB] p.88). Namely, the natural numbers are usually axiomatized with the following (Dedekind) Peano's axioms characterizing the successor function S on N:

(S0) $(\exists!m)m\in(N\setminus S(N))$

(Si) $S(m) = S(n)\longrightarrow m = n$

(SI) $(N\setminus S(N))\subseteq M\ \&\ S(M)\subseteq M\longrightarrow M = N$.

The weakening consists in the elimination of the injectivity axiom for S. Devidé was able to prove the particular axiom (Si) from (S0) and his own (D). Notice that Peano's (SI) is an axiom of induction and it is clear that Devidé's axiom (D) is also some kind of induction axiom. But it must be a stronger induction axiom, because it is well known that (Si) is not derivable from (S0) and (SI). Namely, S defined on $N = \{0,1\}$, with $S(0) = S(1) = 1$, satisfies (S0) and (SI) but does not satisfy (Si), cf. [H] p. 324.

Nevertheless, it appears that the strenght of (D) has not been well understood. We prove that (D) is equivalent to the axiom of transfinite induction, i.e. it is nothing but the statement that the function S, viewed as a binary relation S on N, is well founded. The connection between the relation and the function is the following one: mSn iff $m = S(n)$.

Definition 1. A binary relation S defined on a set N is well founded iff for every nonempty M contained in N

$M \neq \emptyset \rightarrow (\exists n)(n \in M \ \& \ \neg(\exists m \in M) n S m)$,

in other words, iff every M has an S-minimal element.

It is well known that S is well founded on N if there is no infinite sequence of elements of N such that $m_1 S m_2 S m_3 S \ldots$. On the other hand $S(M) = \{n : (\exists m \in M) n S m\}$, so the well foundedness condition (SWF) can be put is the form:

If $M \neq \emptyset$ then $(\exists n)(n \in M \ \& \ n \notin S(M))$ i.e. $(\exists n)(n \in M \setminus S(M))$ i.e. $M \not\subseteq S(M)$.

It follows that relation S defined on N is well founded iff for every $M \subseteq N$:

(SWF) $M \subseteq S(M) \rightarrow M = \emptyset$.

It is the content of Devidé's first lemma in [De] that this is equivalent to (D). Namely, if $M \subseteq S(M)$ then $\mathcal{M} = \{M : M \subseteq S(M)\} \neq \emptyset$. The union $\bigcup\mathcal{M}$ still satisfies (1) $(\bigcup\mathcal{M}) \subseteq S(\bigcup\mathcal{M})$ and, since this implies $S(\bigcup\mathcal{M}) \subseteq S(S(\bigcup\mathcal{M}))$, it follows that $S(\bigcup\mathcal{M}) \in \mathcal{M}$, hence (2) $S(\bigcup\mathcal{M}) \subseteq (\bigcup\mathcal{M})$. By (1) and (2) $(\bigcup\mathcal{M}) = S(\bigcup\mathcal{M})$ and it follows from (D) that $\bigcup\mathcal{M} = \emptyset$, which in turn implies $M = \emptyset$. So, (D) implies (SWF) and it is evident that (SWF) implies (D).

We may summarize our results in the following lemma:

Lemma 1. Devidé's axiom (D) is equivalent to the well foundedness principle (SWF).

It is well known that S defined on N is well founded iff it satisfies the principle of transfinite induction on N:

$(\forall m)((\forall n)(mSN \rightarrow n \in M) \rightarrow m \in M) \rightarrow M=N$ i.e.

$(\forall m)((\forall n)(n \in S^{-1}(m) \rightarrow n \in M) \rightarrow m \in M) \rightarrow M=N$ i.e.

(STI) $(\forall m)(S^{-1}(m) \subseteq M \rightarrow m \in M) \rightarrow M=N$,

or in other words, iff for every S-inductive subset M, M=N.

Remark: We can reformulate (STI) in the followwing way:

$(\forall m)((\forall n)(mSn \rightarrow n \in M) \rightarrow m \in M) \rightarrow M=N$ iff

$(\forall m)(m \notin M \rightarrow (\exists n)(mSn \,\&\, n \in M)) \rightarrow M=N$ iff

$(\forall m)(m \in \{k:k \notin M\} \rightarrow m \in S(\{k:k \notin M\})) \rightarrow M=N$ iff

(STI') $\{k:k \notin M\} \subseteq S(\{k:k \notin M\}) \rightarrow M=N$.

It is quite trivial to prove the equivalence of well foundedness in form (SWF) and transfinite induction in form (STI').

We have the following corollary:

Corollary 1. Devidé's axiom (D) is equivalent to the transfinite induction principle (STI).

Now, it is easy to see where the source of the strenght of (D) comes from. The axiom (D) is tronger then (SI), in the same way in which the transfinite induction is stronger then the (normal) induction.

Of course, in presence of (Si) it is easy to prove that (SI) and (STI) are equivalent. Namely, if we suppose that S is injective (i.e. $mSp \,\&\, mSq \rightarrow p=q$) then $S^{-1}(m)$ is a singleton for every $m \in S(N)$ and empty for every $m \in N \setminus S(N)$. Hence (STI) is equivalent to

$[(\forall m \in N \setminus S(N))(m \in M)\ \&\ (\forall m \in S(N))(S^{-1}(m) \in M \rightarrow m \in M)] \rightarrow M=N$ i.e.

$$N \setminus S(N) \subseteq M\ \&\ S(M) \subseteq M — M=N$$

which is (SI). We have the following lemma.

Lemma 2. Let S be an injective binary relation on a set N. Then (STI) iff (SI). (Note that we presuppose nothing about functionality of S.)

It was already mentioned that (even for a functional relation S) injectivity axiom (Si) is not derivable from (SO) and (SI), whereas Devidé proved that (Si) is derivable from (SO) and (D). Devidé's proof employs Dedekind's technique of forming the "Kette" of an element in N, cf. [D], which itself is only a variant of Frege's "Reihe" of an element, cf. [F]. It was observed in [FS] that this technique may be replaced by making use of the fact that the ordering of N is the transitive closure of S. Felscher-Schmidt's technique may be used to prove Devidé's result.

Definition 2. The transitive closure $\bar{S}$ of a relation S on a set N is the smallest transitive relation on N which contains S.

It is well known that $m\bar{S}n$ iff there wxist a finite (possibly empty) sequence (p,...,q) of elements of N such that $mSp...qSn$. It follows immediately that transitive closure $\bar{S}$ of a well founded relation S is also a well founded relation. Namely, if $\bar{S}$ is not well founded then there is an infinite sequence $m_1\bar{S}m_2\bar{S}m_3...$. It follows that $m_1S...Sm_2S...Sm_3...$ i.e. S itself is not well founded. Hence we have the following lemma.

Lemma 3. Let S be a binary relation on a set N. If S is well founded then $\bar{S}$ is also well founded.

If a well founded relation S satisfies (SO) and if it is also functional, i.e. pSn and qSn implies p=q, then every m is $\bar{S}$-comparable with every n, i.e. $m\bar{S}n$ or m=n or $n\bar{S}m$. This is the content of the following lemma.

Lemma 4. Let S be a functional well founded relation which satisfies (SO) on N. Then every $m \in N$ is $\bar{S}$-comparable with every $n \in N$.

Proof. Let k be any element of N. We have to prove that $m \in C_k = \{p : p$ is $\bar{S}$-comparable with $k\}$ for every $m \in N$. S satisfies (STI), hence it is enought to prove that $m \in C_k$ under the assumption that $S^{-1}(m) \subseteq C_k$.

If $S^{-1}(m) \neq \emptyset$ then there is p such that mSp and we may suppose that $p \in C_k$, i.e. we may suppose that (i) p=k or (ii) $p\bar{S}k$ or (iii) $k\bar{S}p$. If (i) or (ii) holds then mSp implies $m\bar{S}k$, that is $m \in C_k$. If (iii) holds then kS...uSp and mSp implies m=u, by functionality of S. Hence $k\bar{S}m$, that is $m \in C_k$ again.

If $S^{-1}(m) = \emptyset$ then (SO) implies that m is the unique element of N with this property. As usual we denote it with 0. We have to prove that $0 \in C_k$, for every $k \in N$. By (STI), it is enought to prove that $0 \in C_k$ under the assumption that $0 \in C_p$ for $p \in S^{-1}(k)$. If $S^{-1}(k) \neq \emptyset$ then there is p such that kSp and we may suppose that $0 \in C_p$, i.e. (i) 0=p or (ii) $0\bar{S}p$ or (iii) $p\bar{S}0$. The same argument as above proves that $0 \in C_k$ in each of the cases. If $S^{-1}(k) = \emptyset$ then k=0 and it is trivialy true that $0 \in C_k$.

Now, we can prove Devidé's theorem.

<u>Devidé's Theorem</u> If S is functional well founded relation which satisfies (SO) on N, then S is injective.

<u>Proof</u> Notice first that it is trivially true that any well founded relation is irreflexive. Now, let us suppose that S is not injective. Then there are $m,p,q \in N$ such that $p \neq q$ & $m\bar{S}p$ & $m\bar{S}q$. By lemma 4. it follows from $p \neq q$ that $p\bar{S}q$ or $q\bar{S}p$. Let us suppose $p\bar{S}q$. Then pSq or there is v such that pS... ...vSq. If pSq then .m=p follows from mSq and functionality of S in contradiction with mSp (S is irreflexive). If pS... ...vSq then similarly m=v, and because of $p\bar{S}v$ it follows $p\bar{S}m$. Together with mSp it gives mSm, a contradiction again (because $\bar{S}$ is a well founded relation by lemma 3.).

We may conclude that Devidé's theorem follows from lemmae 3. and 4, i.e. from well foundedness of $\bar{S}$ and $\bar{S}$-comparability of elements of N. From these same properties of $\bar{S}$ the irreflexivity of $\bar{S}$ and the transitivity of $\bar{S}$ can be concluded. We know that irreflexivity is an immediate consequence of well foundednes. Suppose now that $m\bar{S}n$ & $n\bar{S}p$. By comparability of $\bar{S}$, m=p or $p\bar{S}m$ or $m\bar{S}p$. If m=p then $m\bar{S}n\,\bar{S}m\,\bar{S}n \ldots$; if $p\bar{S}m$ then $m\bar{S}n\,\bar{S}p\,\bar{S}m\,\bar{S}n\,\bar{S}p \ldots$, both contrary to the well foundednes of $\bar{S}$. Hence $m\bar{S}p$, i.e. $\bar{S}$ is transitive. In other words $\bar{S}$ is a well order on an infinite set N with a unique limit number (cf. (SO)). But here we have an axiomatization of natural numbers. It characterize order relation $\bar{S}$ and it's domain N, with the following axioms:

$(\bar{S}0)$ $(\exists! m) m \in (N \setminus S(N))$

$(\bar{S}\infty)$ $(\forall n)(\exists m) m \bar{S} n$

$(\bar{S}C)$ $m = n \vee m\bar{S}n \vee n\bar{S}m$

$(\bar{S}WF)$ $M \subseteq \bar{S}(M) \rightarrow M = \emptyset$

<u>Remark:</u> S in $(\bar{S}0)$ is defined in the following way:

$$mSn \text{ iff } m\bar{S}n \;\&\; \neg(\exists k)(m\bar{S}k \;\&\; k\bar{S}n).$$

$(\bar{S}0)$ asserts that there is only one limit number, i.e. number without an immediate predecassor. $(\bar{S}\infty)$ is an infinity axiom. Comparability $(\bar{S}C)$ and well foundedness $(\bar{S}WF)$ imply that $\bar{S}$ is a well order on N (cf. above). On the other hand S-axiomatization has the following axioms:

$(S0)$ $(\exists! m) m \in N \setminus S(N)$

$(S\infty)$ $(\forall n)(\exists m) m S n$

(Sf) $pSn \;\&\; qSn \rightarrow p = q$

(SWF) $M \subseteq S(M) \rightarrow M = \emptyset.$

(S0) again asserts that there is only one number without any immediate predeccessors. $(S\infty)$ is again an infinity axiom. (Sf) asserts that S is functional, (SWF) that S is well founded. It follows that S is injective.

Both relations, S and $\bar{S}$, are actually successors: the functional relation S is the immediate successor while the ordering relation $\bar{S}$ is the general successor $>$. It is interesting that they have three common defining properties (i. e. axioms): (0), (∞) and (WF), and only one specific: functionality for S and comparability for $>$. Hence, the natural numbers are axiomatized with the following axioms:

(Σ0) $(\exists! m) m \in N \setminus \Sigma_o(N)$,

(Σ∞) $(\forall m)(\exists n) m\Sigma n$,

(ΣWF) $M \subseteq \Sigma(M) \longrightarrow M = \emptyset$.

Remark: Σ_o in (Σ0) is defined in the following way:

$m\Sigma_o n$ iff $m\Sigma n$ & $\neg(\exists k)(m\Sigma k$ & $k\Sigma n)$.

If we axiomatize the immediate successor Σ=S we have to add the functionality axiom:

(Sf) pSm & $qSn \longrightarrow p = q$.

If we axiomatize the general successor $\Sigma = >$ we have to add the comparability axiom:

(>C) $m = n \vee m > n \vee n > m$.

If S is primitive, $>$ is definable as $> = \overline{S}$. If $>$ is primitive, S is definable with mSn iff $m > n$ & $\neg(\exists k)(m > k$ & $k > n)$.

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