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ON THE EQUIVALENCE OF THE SOLUTION LEMMA AND ACZEL'S ANTIFOUNDATION AXIOM

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We prove, in a simpler way (cf. Aczel's proof in [A], pp. 13-16), that Aczel's anti-foundation axiom AFA implies the solution lemma (for unfounded sets), but we also prove that the solution lemma implies AFA. Our working theory is an extension of ZFC^- (ZFC minus the foundation axiom), which admits individuals (i.e. atoms) and proper classes. These amendments make the first proof simpler and the second one almost immediate.

The extended theory has the membership relation $\in$ as its only undefined relation and the empty set $\varnothing$ as its only undefined object (it is not possible to define $\varnothing$ in a theory which admits individuals). We use capitals A, B, C, ... for any objects of the theory: individuals, sets or proper classes. Elements (i.e. individuals or sets) are defined by

$$\mathrm{el}A \leftrightarrow \exists B(A\in B).$$

Classes (i.e. sets or proper classes) are defined by

$$\mathrm{cl}A \leftrightarrow \exists B(B\in A) \vee A=\varnothing.$$

Individuals are defined as non-classes,

$$\mathrm{ind}A \leftrightarrow -\mathrm{cl}A.$$

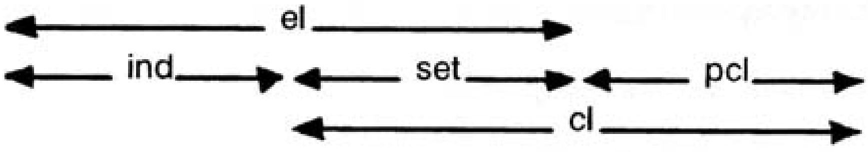
Proper classes are defined as non-elements,

$$\mathrm{pcl}A \leftrightarrow -\mathrm{el}A.$$

Sets are defined as classes which are elements,

$$\mathrm{set}A \leftrightarrow \mathrm{el}A \ \&\ \mathrm{cl}A.$$

Hence, sets are classes small enough to be elements of other classes.



To make our formulae shorter we use small letters $x,y,z,\ldots$ for elements and capitals $X,Y,Z,\ldots$ for classes.

The axioms of the extended theory are:

(EA) **Extensionality:** $Y\subseteq Z \;\&\; Z\subseteq Y \rightarrow Y=Z$,

where $Y\subseteq Z$ means $\forall x(x\in Y \rightarrow x\in Z)$.

(CSA) **Comprehension schema:** $\exists X\forall z(z\in X \leftrightarrow \Psi(z))$,

where $\Psi(z)$ may be any formula of the theory which does not contain X.

These two axioms guarantee that there exists the unique class $\{z\colon \Psi(z)\}$ Hence, there exist the following classes:

$\{x,y\} = \{z\colon z=x \vee z=y\}$

$PX = \{z\colon z\subseteq X\}$

$UX = \{z\colon \exists y(z\in y \;\&\; y\in X)\}$

$\bigcap X = \{z\colon \forall y(y\in X \rightarrow z\in y)\}$

$\omega = \bigcap\{z\colon 0\in z \wedge \forall x(x\in z \rightarrow x^{+}\in z\}$, where $x^{+} = xU\{x\}$.

$Ind = \{x\colon ind\ x\}$

$V[Ind] = \{x\colon set\ x\}$

Class V[Ind] is the universe of all sets constructed over the set of all individuals Ind. By Russell's argument of all sets not belonging to themselves, we my prove that V[Ind] is a proper class. On the other hand, each element of V[Ind] is a set (i.e. it is small enough to be an element).

The next axioms assert setness, i.e. smallness, of some fundamental classes:

(OA) **The empty set:** $\varnothing\in V[Ind]$

(2A) **Pair:** $\forall x\forall y(\{x,y\}\in V[Ind])$

(∞A) **Infinity:** $\omega\in V[Ind]$

(UA) **Union:** $\forall x(Ux\in V[Ind])$

(PA) **Powerset:** $\forall x(Px\in V[Ind])$

Infinity axiom (∞A) demands that the set universe V should be greater than V_ω, but it does not demand that V should be greater then $V_{\omega\cdot 2}$ (cf. below what is V_α, for any ordinal α). To guarantee this, one more axiom is needed:

(RA) **Replacement:** $\mathrm{func}X \ \&\ \mathrm{dom}X \in V[\mathrm{Ind}] \rightarrow \mathrm{ran}X \in V[\mathrm{Ind}]$.

If a function has a small domain it has to have a small range (where "function", "domain" and "range" are defined in the standard way).

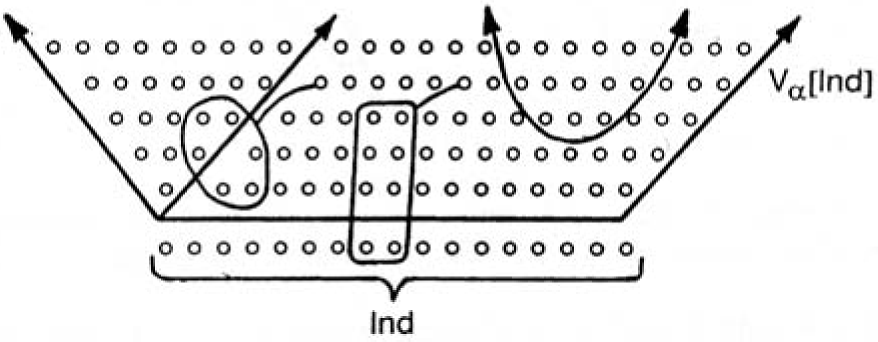
The axioms (EA), (CSA), ($\varnothing$A), (2A), (∞A), (UA), (PA) and (RA), together with the unavoidable axiom of choice (CA), constitute our **theory of sets, individuals and proper classes**. From these axioms we can develop the standard theory of ordinal numbers, so as to define the proper class of all ordinals, $O = \{x\colon x \text{ is an ordinal}\}$, and to prove the principle of transfinite recursion on ordinals. According to the principle there is a unique function from O to V[Ind], with $\alpha \mapsto V_\alpha[\mathrm{Ind}]$, such that

$$V_\varnothing[\mathrm{Ind}] = \varnothing \quad \text{and} \quad V_\alpha[\mathrm{Ind}] = P((\bigcup_{\beta<\alpha} V_\beta[\mathrm{Ind}]) \cup \mathrm{Ind}).$$

Hence, it is possible to define the universe of all founded sets

$$V_O[\mathrm{Ind}] = \bigcup\{V_\alpha[\mathrm{Ind}]\colon \alpha \in O\}.$$

According to the definition, $V_O[\mathrm{Ind}]$ has the following structure.



At the bottom there are individuals (which belong neither to $V_O[\mathrm{Ind}]$ nor to the whole universe of sets V[Ind], because individuals are not sets).

All sets whose elements are individuals or belong to levels below a certain level α constitues $V_\alpha[\mathrm{Ind}]$. They are represented by points at level α. Hence to every point at level α there corresponds a closed domain of points below this level.

The hierarchy of $V_\alpha[\mathrm{Ind}]$ is cumulative:

$$\alpha\leq\beta\rightarrow V_\alpha[\mathrm{Ind}]\subseteq V_\beta[\mathrm{Ind}]$$

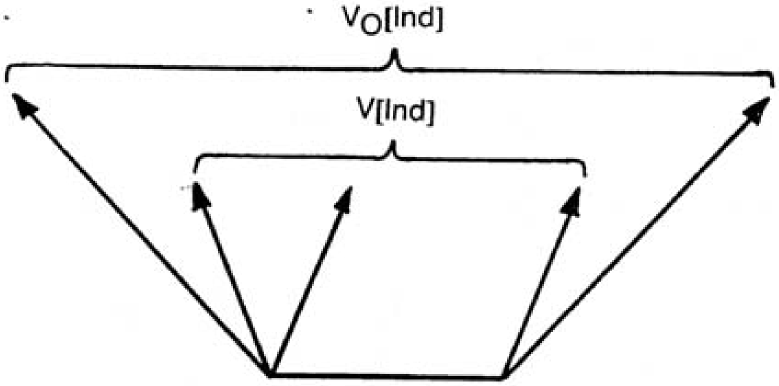
and the union of all $V_\alpha[\mathrm{Ind}]$ is $V_O[\mathrm{Ind}]$, the universe of all founded sets. If we start with no individuals, we get the universe of pure founded sets V_O, which is part of any $V_O[\mathrm{Ind}]$. This part is represented by the cone on the left.

The proper classes are classes which penetrate through all levels. They are represented by open domains (V_O and $V_O[\mathrm{Ind}]$ are two examples, also). Hence, there are no points to represent them at any level α.

(FA) **Foundation axiom:** $\forall X\exists y(y\in X \;\&\; y\cap X=\varnothing)$,

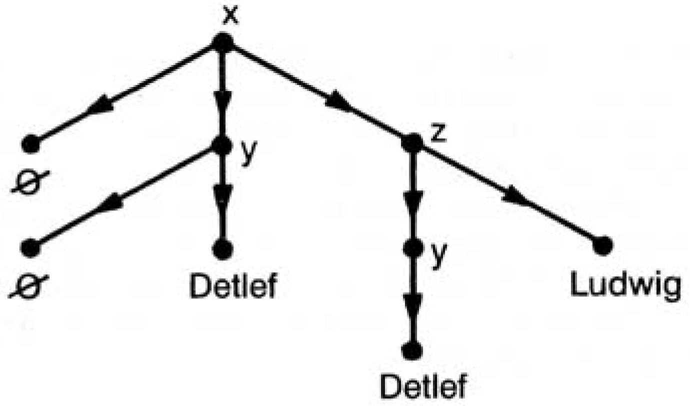
looks very simple, although its real content is: There are no other sets but founded sets which constitute $V_O[\mathrm{Ind}]$, i.e. $V_O[\mathrm{Ind}]=V[\mathrm{Ind}]$. This is Cantor's conception of set. Accordingly, its theory is axiomatized with (EA), (CSA), ($\varnothing$A), (2A), (∞A), (UA), (PA), (RA), (CA) and (FA).

Aczel's conception is broader. It denies (FA). There are unfounded sets, i.e. $V_O[\mathrm{Ind}]\subset V[\mathrm{Ind}]$.

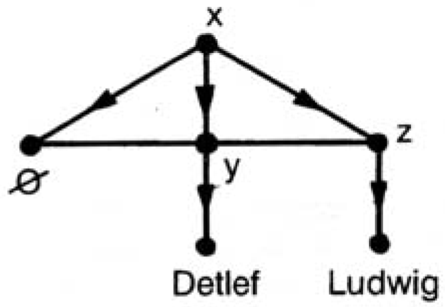


The broader universe of (founded and unfounded) sets is characterized by Aczel's antifoundation axiom (AFA). Let us introduce it as briefly as possible.

Consider the ordinary set $x = \{\varnothing,y,z\}$, where $y = \{\varnothing,\text{Detlef}\}$ and $z = \{y,\text{Ludwig}\}$. The best way to picture this set is with the following labelled graph. (As usual, a **graph** is a class of **nodes** and a class of **arrows**. Any class G can be a class of nodes, and any subclass of $G\times G$ can be a class of arrows. If (x,y) is an arrow it is customary to write $x\rightarrow y$ and say that y is a **child** of x. A node with no arrow starting from it is said to be **childless**. A **path** is a finite or infinite sequence of nodes linked by arrows: $x\rightarrow y\rightarrow z\rightarrow\ldots$. A graph is **founded** if it has no infinite path. A labelled graph is a graph with a **label function** f defined on its nodes. In the following example the value of f on the rightmost node is Ludwig.)

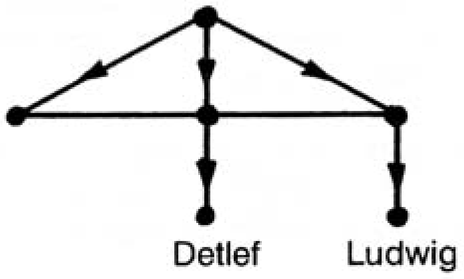


The arrows in such a graph represents the converse membership relation (think of ← as prolonged ∈). Detlef, Ludwig and the empty set have no elements (Detlef and Ludwig are individuals), and so they label the childless nodes. One and the same class may well be pictured by many different graphs. Consider, for example, the following graph.



In the same way any class can be depicted by a graph. As a matter of fact any class X is depicted by (1) the class of nodes consisting of X (if X is a set, i.e. a "point"), elements of X, elements of elements of X, etc. and (2) the class of arrows coinciding with the converse membership relation restricted to the class of nodes.

Conversely, let us start with the sterile labelled graph in which only (some of the) childless nodes are labelled



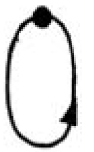
There is only one way to extend the sterile label function to all nodes in such a way that the elements of the set assigned to a node are the elements (i.e. individuals or sets) assigned to the children of that node. A label function with this property is said to be a **decoration** of the graph. Hence, there is a unique decoration of our sterile labelled graph. This is no accident. The simple consequence of the principle of transfinite recursion on founded relations is that every founded sterile labelled graph has a unique decoration which extends its sterile label function (this is Mostowski collapsing lemma modified so as to admit individuals and proper classes). So, there is a perfect match between the universe of founded sterile labelled graphs and the universe of founded sets. Aczel's conception is that there is a perfect match between the universe of sterile labelled graphs (founded or not) and the universe of all sets (founded or not). This is the content of Aczel's antifoundation axiom (modified so as to admit individuals and proper classes).

> (AFA) **Antifoundation:** Every sterile labelled graph has a unique decoration which extends its sterile labelled function.

More precisely, if (G,A,s) is a sterile labelled graph with (1) class of nodes G, (2) class of arrows A and (3) sterile label function $s:S\rightarrow Ind$, where S is a subset of childless nodes of G and Ind is a set of individuals, then there is a unique function $d:G\rightarrow V[Ind]\cup Ind$ such that

$$d(a) = \begin{cases} s(a) & \text{for } a\in S \\ \{d(b):\ b\leftarrow a\} & \text{for } a\notin S. \end{cases}$$

Of course, unfounded graphs (e.g. graphs with cycles) cannot represent founded classes. For example, in Aczel's univers of sets there is a set $\Omega=\{\Omega\}$, because we can represent the membership relation on Ω with the following graph



$$\Omega = \{\Omega\}$$

Moreover, on Aczel's conception this graph unambiguously represents a set. There is only one set Ω equal to its own singelton. (As before, one and the same set may well be represented by many different graphs. Ω is represented by any graph with no childless nodes, cf. [A].)

One of the most fruitful consequences of AFA is the solution lemma. It allows us to assert that sets of all kinds exist even if we do not first represent them with graphs. Namely, any system of equations in indeterminates x, y, z,..., say

$$x = c_x(x,y,z,...)$$

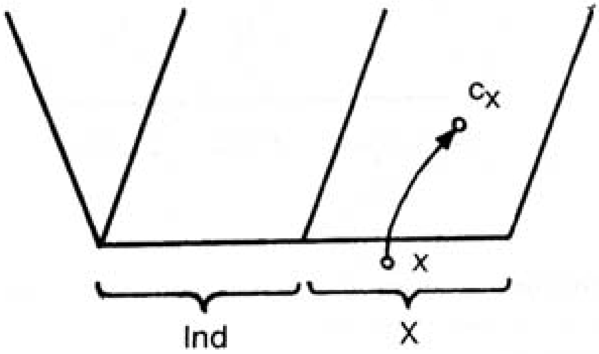
$$y = c_y(x,y,z,...)$$

$$z = c_z(x,y,z,...)$$

$$\vdots$$

has a unique solution in Aczel's universe. We formulate the solution lemma more precisely.

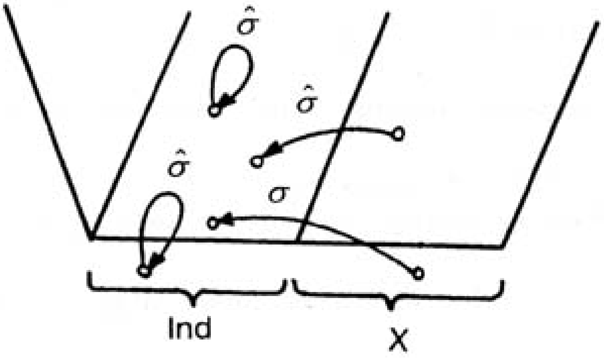
A system of equations in a set of indeterminates X, such that $X \cap \mathrm{Ind} = \emptyset$ and members of X are individuals, is a function $c: X \rightarrow V[\mathrm{Ind} \cup X] \cup \mathrm{Ind}$. To every $x \in X$ there corresponds $c_x \in V[\mathrm{Ind} \cup X] \cup \mathrm{Ind}$ and we write $x = c_x$, $x \in X$, to indicate the system of equations.



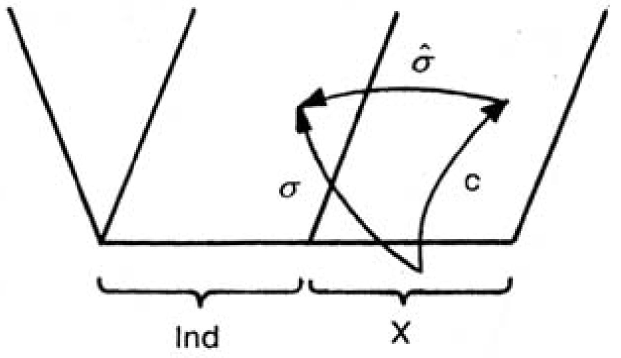
A solution to the system of equations $c: X \rightarrow V[\mathrm{Ind} \cup X] \cup \mathrm{Ind}$ is a function $\sigma: X \rightarrow V[\mathrm{Ind}] \cup \mathrm{Ind}$ such that $\sigma = \hat{\sigma} \circ c$, where $\hat{\sigma}$ is a unique function from the following lemma.

Substitution lemma:
For each $\sigma: X \rightarrow V[\mathrm{Ind}] \cup \mathrm{Ind}$ there is a unique function $\hat{\sigma}: V[\mathrm{Ind} \cup X] \cup \mathrm{Ind} \rightarrow V[\mathrm{Ind}] \cup \mathrm{Ind}$ such that

$$\hat{\sigma}(a) = \begin{cases} a & \text{for } a \in \mathrm{Ind} \\ \{\hat{\sigma}(b) : b \in a \ \& \ b \in V[\mathrm{Ind} \cup X]\} \cup \{\sigma(x) : x \in a \ \& \ x \in X\} & \text{for } a \in V[\mathrm{Ind} \cup X]. \end{cases}$$



Solution lemma:
For each system of equations $x=c_x$, $x\in X$, there is a unique solution σ such that $\sigma=\hat{\sigma}\circ c$.



Proof of the substitution lemma: Consider the sterile labelled graph with:
(1) $V[\mathrm{Ind}\cup X]\cup\mathrm{Ind}$ as its class of nodes,
(2) the converse membership relation $\in$ extended with $\sigma x\leftarrow a$ iff $x\in a$ & $x\in X$, as its class of arrows and
(3) the identity function id: $\mathrm{Ind}\rightarrow\mathrm{Ind}$, as its sterile label function.

By AFA (in Aczel's universe or by FA in Cantor's founded universe) there is a unique decoration $d:V[\mathrm{Ind}\cup X]\cup\mathrm{Ind}\rightarrow V[\mathrm{Ind}]\cup\mathrm{Ind}$ such that

$$d(a)=\begin{cases}\mathrm{id}(a)=a & \text{for } a\in\mathrm{Ind}\\ \{d(b):b\in a\ \&\ b\in V[\mathrm{Ind}\cup X]\}\cup\{d(\sigma(x)):\sigma(x)\leftarrow a\} & \text{for } a\in V[\mathrm{Ind}\cup X].\end{cases}$$

The decoration d restricted to V[Ind] is an identity (because the identity function is a decoration of V[Ind] and d is the unique decoration). Hence, $d(\sigma(x))=\sigma(x)$ for every $\sigma(x)$, because $\sigma(x)\in V[\mathrm{Ind}]$. It follows that

$$d(a)=\{d(b):\ b\in a\ \&\ b\in V[\mathrm{Ind}\cup X]\}\ \cup\ \{d(\sigma(x)):\sigma(x)\leftarrow a\}=$$

$$=\{d(b):\ b\in a\ \&\ b\in V[\mathrm{Ind}\cup X]\}\ \cup\ \{\sigma(x):x\in a\ \&\ x\in X\},$$

for $a\in V[\mathrm{Ind}\cup X]$. The sought for $\hat{\sigma}$ is d.

Proof of the solution lemma from AFA: Consider the sterile labelled graph with:
(1) $V[\mathrm{Ind}\cup X]\cup\mathrm{Ind}$ as its class of nodes,
(2) the converse membership relation $\in$ extended with $c_x\leftarrow a$ iff $x\in a$ & $x\in X$, as its class of arrows,
(3) the identity function id: $\mathrm{Ind}\rightarrow\mathrm{Ind}$, as its sterile label function.

By AFA there is a unique decoration d: $V[\mathrm{Ind}\cup X]\cup\mathrm{Ind}\rightarrow V[\mathrm{Ind}]\cup\mathrm{Ind}$ such that

$$d(a) = \begin{cases} id(a)a & \text{for } a \in Ind \\ \{d(b): b \in a \ \& \ b \in V[Ind \cup X]\} \cup \{d(c_X): c_X \leftarrow a\} & \text{for } a \in V[Ind \cup X]. \end{cases}$$

If we define $\sigma(x)$ by

$$\sigma(x) = d(c_X)$$

it follows immediately that $d=\hat{\sigma}$, i.e.

$$\sigma = \hat{\sigma} \circ c.$$

Proof of AFA from the solution lemma: Let (X,A,s) be sterile labelled graph with (1) X as its class of nodes, (2) A as its class of arrows,(3) $s:S \rightarrow Ind$ as its sterile label function, where S is a subset of childless nodes of X and Ind is a set of individuals, such that $Ind \cap X = \varnothing$.
We have to prove that there is a unique function $d: X \rightarrow V[Ind] \cup Ind$ such that

$$d(x) = \begin{cases} s(x) & \text{for } x \in S \\ \{d(y): y \leftarrow x\} & \text{for } x \notin S. \end{cases}$$

Consider the system of equations $c: X \rightarrow V[Ind \cup X] \cup Ind$ defined by

$$c_X = \begin{cases} s(x) & \text{for } x \in S \\ \{y: y \leftarrow x\} & \text{for } x \notin S. \end{cases}$$

By the solution lemma there is a unique function $\sigma: X \rightarrow V[Ind] \cup Ind$ such that $\sigma(x) = \hat{\sigma}(c_X)$. If $x \in S$ then

$$\sigma(x) = \hat{\sigma}(c_X) = \hat{\sigma}(s(x)) = s(x),$$

because $s(x) \in Ind$. If $x \notin S$ then

$$\sigma(x) = \hat{\sigma}(c_X) = \{\hat{\sigma}(b): b \in c_X \ \& \ b \in V[Ind \cup X]\} \cup \{\sigma(y): y \leftarrow x\}.$$

But c_X has no members from $V[Ind \cup X]$, so it follows that

$$\sigma(x) = \{\sigma(y): y \leftarrow x\}.$$

The sought for d is σ.

Note that we used only the restricted version of the solution lemma. The version asserts the existence of a unique solution for systems of equations of the first rank:

$$c : X \rightarrow V_1[\mathrm{Ind} \cup X] \cup \mathrm{Ind}.$$

We define the rank of the system of equations $x=c_x$, $x \in X$, as the supremum of the ranks of c_x, $x \in X$. (Intuitively, the rank of c_x is "the maximal number of nested braces of c_x". For example, {{Detelf},{0,{Ludwig}}} has rank 3.) It follows that any system of equations can be reduced to a system of the first rank.

[A] Aczel, P. *Non-well-founded sets.* CSLI Lecture notes No. 14 (1988).

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