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ON THE LOGICAL SYSTEM L_1

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We have recently introduced the logical system L_1 (cf. [3]). L_1 is the intuitionistic system LI (cf. [1]) extended with schema

$$S = (\forall x)\,(A(x) \vee \neg A(x)) \to (\forall x)\,A(x) \vee (\exists x)\,\neg A(x),$$

i.e. $L_1 = LI + S$. The purpose of this paper is to compare L_1 with some other systems.

It is well known that LI + MP, where

$$MP = (\forall x)\,(A(x) \vee \neg A(x)) \to \neg(\forall x)\,A(x) \supset (\exists x)\,\neg A(x),$$

properly extends LI and also that LK properly extends LI + MP, i.e.

$$LI < LI + MP < LK.$$

LI + MP is (formally) the logical basis of Russian constructivism, and so we need not comment on it's interest.

We will also consider LI + $T_\forall$, where

$$T_\forall = (\forall x)\,(A(x) \vee \neg A(x)) \to (\forall x)\,A(x) \vee \neg(\forall x)\,A(x),$$

as well as LI + $T_\exists$, where

$$T_\exists = (\forall x)\,(A(x) \vee \neg A(x)) \to (\exists x)\,A(x) \vee \neg(\exists x)\,A(x),$$

and finaly LI + T, where

$$T = T_\forall \wedge T_\exists.$$

These systems are interesting because of the following Corollary 1 to Theorem 1:

Corollary 1. *If an axiomatic theory (with axioms $\mathscr{A}$), which is logically based on* LI + T *(and which we will denote* (LI + T) $\mathscr{A}$*), has only decidable atomic predicates, then all the predicates of the theory are decidable. It means that the axiomatic theory* (LI + T) $\mathscr{A}$ *(which is logically based on* LI + T*) is the same theory as the axiomatic theory* (LK) $\mathscr{A}$ *(which is based on* LK*), provided that all the atomic predicates of $\mathscr{A}$ are decidable.*

Remark. $P(x, \ldots)$ is *decidable* (in a theory) if $(\forall x) \ldots (P(x, \ldots) \vee \neg P(x, \ldots))$ is provable (in the theory).

Example. The atomic predicates of the first order Peano's axioms A are decidable. Hence (LI + T) A = (LK) A = PA.

Theorem 1. *Every sequent of the form $T_\wedge$, $T_\vee$, $T_\supset$ and $T_\neg$ is provable in* LI, *where*

$$T_\wedge = A \vee \neg A, B \vee \neg B \to (A \wedge B) \vee \neg(A \wedge B),$$
$$T_\vee = A \vee \neg A, B \vee \neg B \to (A \vee B) \vee \neg(A \vee B),$$
$$T_\supset = A \vee \neg A, B \vee \neg B \to (A \supset B) \vee \neg(A \supset B),$$
$$T_\neg = A \vee \neg A \to \neg A \vee \neg\neg A.$$

Proof.

$T_\wedge$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \wedge B \to A}}{\neg A, A \wedge B \to}}{\neg A \to \neg(A \wedge B)}}{\neg A \to (A \wedge B) \vee \neg(A \wedge B)} \quad \dfrac{\dfrac{A \to A \quad B \to B}{A, B \to A \wedge B}}{A, B \to (A \wedge B) \vee \neg(A \wedge B)}}{A \vee \neg A, B \to (A \wedge B) \vee \neg(A \wedge B)} \quad \dfrac{\dfrac{\dfrac{\dfrac{B \to B}{A \wedge B \to B}}{\neg B, A \wedge B \to}}{\neg B \to \neg(A \wedge B)}}{\neg B \to (A \wedge B) \vee \neg(A \wedge B)}}{A \vee \neg A, B \vee \neg B \to (A \wedge B) \vee \neg(A \wedge B)}$$

$T_\vee$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A \to A \vee B}}{A \to (A \vee B) \vee \neg(A \vee B)} \quad \dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A, \neg A \to} \quad \dfrac{B \to B}{B, \neg B \to}}{A \vee B, \neg A, \neg B \to}}{\neg A, \neg B \to \neg(A \vee B)}}{\neg A, \neg B \to (A \vee B) \vee \neg(A \vee B)}}{A \vee \neg A, \neg B \to (A \vee B) \vee \neg(A \vee B)} \quad \dfrac{\dfrac{B \to B}{B \to A \vee B}}{B \to (A \vee B) \vee \neg(A \vee B)}}{A \vee \neg A, B \vee \neg B \to (A \vee B) \vee \neg(A \vee B)}$$

$T_\supset$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \to B}{A, B \to B}}{B \to A \supset B}}{B \to (A \supset B) \vee \neg(A \supset B)} \quad \dfrac{\dfrac{\dfrac{A \to A \quad \dfrac{B \to B}{\neg B, B \to}}{A, A \supset B, \neg B \to}}{A, \neg B \to \neg(A \supset B)}}{A, \neg B \to (A \supset B) \vee \neg(A \supset B)}}{A, B \vee \neg B \to (A \supset B) \vee \neg(A \supset B)} \quad \dfrac{\dfrac{\dfrac{\dfrac{A \to A}{A, \neg A \to}}{A, \neg A \to B}}{\neg A \to A \supset B}}{\neg A \to (A \supset B) \vee \neg(A \supset B)}}{A \vee \neg A, B \vee \neg B \to (A \supset B) \vee \neg(A \supset B)}$$

$T_\neg$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}}{A \to \neg\neg A}}{A \to \neg A \vee \neg\neg A} \quad \dfrac{\neg A \to \neg A}{\neg A \to \neg A \vee \neg\neg A}}{A \vee \neg A \to \neg A \vee \neg\neg A}$$

Every sequent of the form $T_\forall$ is provable in LI + $T_\forall$ and every sequent of the form $T_\exists$ is provable in LI + $T_\exists$. Hence, every sequent of the form $T_\wedge$, $T_\vee$, $T_\supset$, $T_\neg$, $T_\forall$ or $T_\exists$ is provable in LI + T, and Corollary 1 follows by induction on the complexity of formulae.

Lemma 1. *Every sequent of the form T is provable in* LI + $T_\exists$, *i.e.* LI + $T_\exists$ = LI + T.

Proof. We have already proved in Theorem 1 that

$$\vdash_{LI} A(a) \vee \neg A(a) \to \neg A(a) \vee \neg\neg A(a).$$

Hence

(1) $$\vdash_{LI} (\forall x)\,(A(x) \vee \neg A(x)) \to (\forall x)\,(\neg A(x) \vee \neg\neg A(x)).$$

From definition of LI + $T_\exists$ follows

(2) $$\vdash_{LI+T_\exists} (\forall x)\,(\neg A(x) \vee \neg\neg A(x)) \to (\exists x)\,\neg A(x) \vee \neg(\exists x)\,\neg A(x).$$

From (1) and (2) it follows

(3) $$\vdash_{LI+T_\exists} (\forall x)\,(A(x) \vee \neg A(x)) \to (\exists x)\,\neg A(x) \vee \neg(\exists x)\,\neg A(x).$$

It is easy to prove

(4) $$\vdash_{LI} (\exists x)\,\neg A(x) \vee \neg(\exists x)\,\neg A(x) \to \neg(\forall x)\,A(x) \vee (\forall x)\,\neg\neg A(x).$$

From (3) and (4) it follows

(5) $$\vdash_{LI+T_\exists} (\forall x)\,(A(x) \vee \neg A(x)) \to \neg(\forall x)\,A(x) \vee (\forall x)\,\neg\neg A(x).$$

It is easy to prove

(6) $$\vdash_{LI} (\forall x)\,(A(x) \vee \neg A(x)) \to (\forall x)\,(A(x) \equiv \neg\neg A(x)).$$

From (5) and (6) there follows our result

$$\vdash_{LI+T_\exists} (\forall x)\,(A(x) \vee \neg A(x)) \to \neg(\forall x)\,A(x) \vee (\forall x)\,A(x).$$

Lemma 2. a) *Every sequent of the form $T_\forall$ is provable in* L_1. b) *Every sequent of the form $T_\exists$ is provable in* L_1.

It means that L_1 extends LI + T, i.e.

$$LI + T \leqq L_1.$$

Proof. a) It is well known that

(7) $$\vdash_{LI} (\exists x)\,\neg A(x) \to \neg(\forall x)\,A(x).$$

From definition of L_1 it follows

(8) $$\vdash_{L_1} (\forall x)\,(A(x) \vee \neg A(x)) \to (\forall x)\,A(x) \vee (\exists x)\,\neg A(x).$$

From (7) and (8) there follows our result

$$\vdash_{L_1} (\forall x)\,(A(x) \vee \neg A(x)) \to (\forall x)\,A(x) \vee \neg(\forall x)\,A(x).$$

b) From definition of L_1 it follows

(9) $$\vdash_{L_1} (\forall x)\,(\neg A(x) \vee \neg\neg A(x)) \to (\forall x)\,\neg A(x) \vee (\exists x)\,\neg\neg A(x).$$

From (9) and (1) it follows

(10) $$\vdash_{L_1} (\forall x)\,(A(x) \vee \neg A(x)) \to (\forall x)\,\neg A(x) \vee (\exists x)\,\neg\neg A(x).$$

It is easy to prove

(11) $$\vdash_{LI} (\forall x)\,\neg A(x) \vee (\exists x)\,\neg\neg A(x) \to \neg(\exists x)\,A(x) \vee (\exists x)\,\neg\neg A(x).$$

From (10) and (11) it follows

(12) $$\vdash_{L_1} (\forall x)\,(A(x) \vee \neg A(x)) \to (\exists x)\,\neg\neg A(x) \vee \neg(\exists x)\,A(x).$$

From (6) and (12) there follows our result

$$\vdash_{L_1} (\forall x)\,(A(x) \vee \neg A(x)) \to (\exists x)\,A(x) \vee \neg(\exists x)\,A(x).$$

Lemma 3. *Every sequent of the form MP is provable in* L_1, *i.e.*

$$LI + MP \leqq L_1.$$

Proof. It is well known that

$$\vdash_{LI} B \vee C \to \neg B \supset C.$$

By appropriate substitution follows

(13) $\vdash_{LI} (\forall x)\,A(x) \vee (\exists x)\,\neg A(x) \to \neg(\forall x)\,A(x) \supset (\exists x)\,\neg A(x).$

Comparing S with MP it is easy to see that (13) proves our result.

Lemma 4. $L_1 = LI + MP + T_\forall$.

Proof. From definition of $LI + MP + T_\forall$ it follows

(14) $\vdash_{LI+MP+T_\forall} (\forall x)\,(A(x) \vee \neg A(x)) \to (\forall x)\,A(x) \vee \neg(\forall x)\,A(x),$

and (cf. MP)

(15) $\vdash_{LI+MP+T_\forall} (\forall x)\,(A(x) \vee \neg A(x)), (\forall x)\,A(x) \vee \neg(\forall x)\,A(x) \to (\forall x)\,A(x) \vee (\exists x)\,\neg A(x).$

From (14) and (15) it follows

(16) $\vdash_{LI+MP+T_\forall} (\forall x)\,(A(x) \vee \neg A(x)) \to (\forall x)\,A(x) \vee (\exists x)\,\neg A(x).$

Hence, every sequent of the form S is provable in $LI + MP + T_\forall$. It means that

(17) $L_1 \leqq LI + MP + T_\forall.$

In Lemma 2 and Lemma 3 we have already proved

(18) $LI + MP + T_\forall \leqq L_1.$

Our result follows from (17) and (18).

We put our results together in the following

Theorem 2.

$$\left.\begin{array}{r} LI + T_\forall \leqq LI + T_\exists = LI + T \leqq \\ LI + MP \leqq \end{array}\right\} LI + T + MP = LI + MP + T_\forall = L_1.,$$

Theorem 3. L_1 *properly extends* $LI + MP$, *i.e.*

$$LI + MP < L_1.$$

Proof. Let A be system of first order Peano's axioms. Let us assume that

(19) $L_1 = LI + MP.$

From (19) it follows

(20) $(L_1)\,A = (LI + MP)\,A.$

In our example we have already proved

(21) $(LI + T)\,A = (LK)\,A.$

We have also proved

(22) $LI + T \leqq L_1 < LK.$

From (21) and (22) it follows

(23) $(\mathrm{LI} + T)\,A = (\mathrm{L}_1)\,A = (\mathrm{LK})\,A$.

From (20) and (23) it follows

(24) $(\mathrm{LI} + MP)\,A = (\mathrm{LK})\,A$.

(24) contradicts the well-known result of KREISEL [2]:

$$(\mathrm{LI} + MP)\,A < (\mathrm{LK})\,A.$$

It follows that our assumption (19) can not be true. Hence (cf. Theorem 2)

$$\mathrm{LI} + MP < \mathrm{L}_1.$$

Corollary 2. LI + *MP does not extend* LI + *T*.

Proof. $\mathrm{LI} + MP < \mathrm{L}_1$. We know from Lemma 4 that $\mathrm{LI} + MP + T = \mathrm{L}_1$. Hence, T is not admissible in LI + *MP*.

Conjecture 1. LI + *T properly extends* LI + $T_\forall$, *i.e.*

$$\mathrm{LI} + T_\forall < \mathrm{LI} + T.$$

Conjecture 2. L_1 *properly extends* LI + *T*, *i.e.*

$$\mathrm{LI} + T < \mathrm{L}_1.$$

Conjecture 3. LI + *T does not extend* LI + *MP*.

Theorem 4. *Conjecture* 2 *is equivalent to Conjecture* 3.

Proof. It follows immediately from Lemma 4.

References

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