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PEACOCK'S PRINCIPLE AND EULER'S EQUATION

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Abstract. The exponentiation is never extended from the real to the complex domain in accordance with Peacock's principle of permanence, although it is the best way of extending the other operations. But we show that we are almost compelled to Euler's equation by Peacock's principle of permanence and also the we are definitely compelled to it if we accept the principle of permanence of differentiability.

C.B. Allendoerfer dedicated his [1] to those authors whose papers on Euler's equation had been rejected by American Mathematical Monthly. He emphasized that the expression $e^{i\omega}$ has to be defined, in order to prove Euler's equation, but his criteria for accepting a definition of the expression as a good one (rigor, simplicity and intuition) are quite vague. We can not be satisfied with such a vague criteria because excellent criteria have existed for a long time. Such is G.Peacock's principle of permanence of equivalent forms announced already in 1833.

A definition of on operation should be extended from a restricted domain to a wider one in such a way as to conserve the crucial algebraic properties of the operation.

The crucial algebraic properties of addition multiplication and exponentiation are as follows

$$\# \begin{cases} a+b=b+a & a\cdot b=b\cdot a \\ (a+b)+c=a+(b+c) & (a\cdot b)\cdot c=a\cdot(b\cdot c) \\ a\cdot(b+c)=a\cdot b+a\cdot c & \\ a^{b+c}=a^b\cdot a^c \qquad (a^b)^c=a^{b\cdot c} & (a\cdot b)^c=a^c\cdot b^c, \end{cases}$$

and the extensions of these operations (from the domain of natural numbers to the domain of complex numbers) were uniquely determined by the principle, in all cases except one. The one with which Euler's equation is concerned.

Mus it be so? Are we compelled by Peacock's principle to define $e^{i\omega}$ as $\cos\omega + i\sin\omega$ (as we are compelled to define $a^{1/n}$ as $\sqrt[n]{a}$ or a^{-n} as $1/a^n$

etc.)? We shall show, that we almost are.

We obtain complex numbers by adding the imaginary unit i to the reals and by combining the old reals with the new unit i using the operations $+$ and $\cdot$ uniquely extended in accordance with Peacock's principle. We immediately realize that any element of the new complex domain is of the form $x+iy$ for real x and y (because of the defining property of i: $i^2=-1$) and that the totality of all new numbers forms a field. But what about exponentiation in the new complex domain? Is it possible to define exponentiation of complex numbers (determined by reals, i, $+$ and $\cdot$) in accordance with Peacock's principle, so as to remain within the complex domain?[1)] We shall show it is.

Notice first that $-i$ has the same defining property as i: $(-i)^2=-1$. So, any calculation with i which ends with the result

$$R(i) = x + iy$$

when performed on $-i$ will end with the result

$$R(-i) = x - iy.$$

But we want to treat exponentiation as a calculation process in the complex domain, so if for real a and ω

$$R(i) = a^{i\omega} = x + iy \qquad \text{then}$$

$$R(-i) = a^{-i\omega} = x - iy.$$

This is also a kind of permanence principle. But then

$$a^{i\omega}\cdot a^{i\omega} = (\text{retaining } \# \text{ by Peacock's principle}^{2)}) = a^{i\omega - i\omega} =$$

$$= a^{o} = 1 = (x+iy)\cdot(x-iy) = x^2+y^2 \qquad \text{i.e.}$$

$$a^{i\omega} = \cos\phi + i\sin\phi .$$

It remains to find out how ϕ depends on a and ω.
$\phi(a,\omega)$ has to be continuous in a and ω if continuity of exponentiation is to be preserved in the complex domain. Hence, the continuity will be presupposed in the sequel. By Peacock's principle we shall in the sequel understand the principle of conservation of continuity and the crucial algebraic properties $\#$.

LEMMA 1. <u>The function</u> $\phi(a,\omega)$ <u>is linear in the second argument</u>:

$$\phi(a,k\cdot\omega) = k\cdot\phi(a,\omega).$$

Proof.

$$\cos\phi(a,\omega_1+\omega_2)+i\sin\phi(a,\omega_1+\omega_2) = a^{i\cdot(\omega_1+\omega_2)} = (\mathrm{Pp}) =$$

$$= a^{i\omega_1}\cdot a^{i\omega_2} = (\cos\phi(a,\omega_1)+i\sin\phi(a,\omega_2))\cdot$$

$$\cdot(\cos\phi(a,\omega_2)+i\sin\phi(a,\omega_2)) = \cos(\phi(a,\omega_1)+ \quad (a,\omega_2)) +$$

$$+ i\sin(\phi(a,\omega_1)+\phi(a,\omega_2)) \qquad \text{i.e.}$$

(1) $$\phi(a,\omega_1+\omega_2) = \phi(a,\omega_1) + \phi(a,\omega_2).$$

Linearity follows from additivity (1) and continuity of ϕ.

LEMMA 2. <u>The function $\phi(a,\omega)$ is linear in the logarithm of the first argument</u>:

$$\phi(a^k,\omega) = k\cdot\phi(a,\omega).$$

Proof.

$$\cos\phi(a_1\cdot a_2,\omega)+i\sin\phi(a_1\cdot a_2,\omega) = (a_1\cdot a_2)^{i\omega} = (\mathrm{Pp}) = a_1^{i\omega}\cdot a_2^{i\omega} =$$

$$= (\cos\phi(a_1,\omega)+i\sin\phi(a_1,\omega))\cdot(\cos\phi(a_2,\omega)+i\sin\phi(a_2,\omega)) =$$

$$= \cos(\phi(a_1,\omega)+\phi(a_2,\omega))+i\sin(\phi(a_1,\omega)+\phi(a_2,\omega)) \qquad \text{i.e.}$$

(2) $$\phi(a_1\cdot a_2,\omega) = \phi(a_1,\omega) + \phi(a_2,\omega).$$

Linearity in logarithm follows from (2) and continuity of ϕ.

If follows from LEMMA 1. that

(3) $$\phi(a,\omega) = k(a)\cdot\omega$$

and from LEMMA 2. that

(4) $$\phi(a,\omega) = \ln a\cdot h(\omega).$$

From (3) and (4) we have

$$k(a)\cdot\omega = \ln a \cdot h(\omega)$$

that is

$$\frac{k(a)}{\ln a} = \frac{h(\omega)}{\omega} \qquad \text{for any } a \text{ and } \omega$$

that is

$$\frac{k(a)}{\ln a} = \frac{h(\omega)}{\omega} = c = \text{const.}$$

Hence

$$\phi(a,\omega) = c\cdot\omega\cdot\ln a\,.$$

So, the only possible definition of exponentiation in the complex domain, which is in accordance with Peacock's principle, is the following one

$$a^{i\omega} = \cos(c\cdot\omega\cdot\ln a) + i\sin(c\cdot\omega\cdot\ln a).$$

It is also easy to see that the crucial algebraic properties # are realy preserved by this definition (for any choice of c).

In particular, we are compelled by Peacock's principle to define

$$e^{i\omega} = \cos(c\cdot\omega) + i\sin(c\cdot\omega),$$

i.e. we are <u>almost</u> compelled to Euler's equation (up to the constant c, which we can choose arbitrarily).

Are we compelled to choose $c=1$ if we want to define exponentiation of complex base with complex exponent in accordance with Peacock's principle? No, we are not:

Let

$$z_1 = r\cdot(\cos\phi + i\sin\phi)$$

and let

$$z_2 = x + iy.$$

Then

$$z_1^{z_2} = (r\cdot(\cos\phi + i\sin\phi))^{(x+iy)} = (Pp) =$$

$$= r^{(x+iy)}\cdot(\cos\phi + i\sin\phi)^{(x+iy)} = (Pp) =$$

$$= r^x \cdot r^{iy} \cdot (\cos\phi + i\sin\phi)^x \cdot (\cos\phi + i\sin\phi)^{iy} =$$

$$= r^x \cdot (\cos(c \cdot y \cdot \ln r) + i\sin(c \cdot y \cdot \ln r)) \cdot (\cos(x \cdot \phi) + i\sin(x \cdot \phi)) \cdot$$

$$\cdot (\cos\phi + i\sin\phi)^{iy} = r^x \cdot (\cos(x \cdot \phi + c \cdot y \ln e) + i\sin(x \cdot \phi + c \cdot y \cdot \ln r)) \cdot$$

$$\cdot (\cos(c \cdot \frac{\phi}{c} \cdot \ln e) + i\sin(c \cdot \frac{\phi}{c} \cdot \ln e))^{iy} =$$

$$= r^x (\cos(x \cdot \phi + c \cdot y \cdot \ln r) + i\sin(x \cdot \phi + c \cdot y \cdot \ln r)) \cdot (e^{i\phi/c})^{iy} =$$

$$= (\mathrm{Pp}) = r^x \cdot e^{-y \cdot \phi/c} \cdot (\cos(x \cdot \phi + c \cdot y \cdot \ln r) + i\sin(x \cdot \phi + c \cdot y \cdot \ln r)),$$

and it is easy to see that the crucial algebraic properties # are preserved by the definition:

$$(r \cdot (\cos\phi + i\sin\phi))^{(x+iy)} = r^x \cdot e^{-y \cdot \phi/c} \cdot (\cos(x \cdot \phi + c \cdot y \cdot \ln r) + i\sin(x \cdot \phi + c \cdot y \cdot \ln r))$$

for any choice of c.

So, Peacock's principle does not compell us to choose (Euler's) $c=1$.

If we add the principle of permanence of diferentiability we are compelled to choose $c=1$. Namely the function $f(z) = a^z$ is diferentiable only for $c=1$. We shall prove this:

The function

$$u + iv = a^{x+iy} =$$
$$= a^x \cdot \cos(c \cdot y \cdot \ln a) + i\, a^x \cdot \sin(c \cdot y \cdot \ln a)$$

is diferentiable only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e. only if

$$a^x \cdot \ln a \cdot \cos(c \cdot y \cdot \ln a) = c \cdot a^x \cdot \ln a \cdot \cos(c \cdot y \cdot \ln a)$$

i.e. only if

$$c = 1.$$

Conclusion. We are almost compelled to Euler's equation by Peacock's principle. We are definitely compelled to it if we also accept the principle of permanence of diferentiability. So, Allendoerfer's condition: $d/d\omega(e^{i\omega}) = i\,e^{i\omega}$, or the Curtiss' condition (cf. p.51): $d/dz(e^z) = e^z$

are unnecesarily strong concerning the diferentiation. Besides, they do not take into consideration the most fundamental principle of permanence - Peacock's principle - which has to remain our guide in extending all the operations, as much as it can.

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1) Notice, that this is not possible for rational numbers. If we define $2^{1/2}$ in accordance with Peacock's principle as $\sqrt[2]{2}$ we do not remain within rationals.

2) In what follows we shall write (for brevity) "Pp" instead of "retaining # by Peacock's principle".

REFERENCES

1. Allendoerfer, C.B., Editorial - The Proof of Euler's Equation, American Mathematical Monthly, vol.55, 1948.

2. Curtiss, D.R., Analytic Functions of a Complex Variable, MAA, 1926.

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