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PREMISS TREE PROOFS AND LOGIC OF CONTRADICTION

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In the present paper a kind of proofs is introduced. They are called *premiss tree proofs* in contrast to common tree proofs, which will be called *conclusion tree proofs*. It is quite easy (and natural) to formulate GOODMAN's "logic of contradiction" (cf. [1]) with the help of premiss tree proofs, and it is hoped that this might make them interesting.

First, we repeat the well-known definition of conclusion tree proofs (cf. [2, p. 44]).

Definition 1. A *conclusion tree proof from a set of formulae X to a set of formulae Y by rules of inferences R* is a finite tree of formulae such that for each file φ (i.e. for each initial segment of a branch)

either (i) rank $\varrho(\varphi)$ (i.e. the set of immediate successors of φ) consists of an occurrence of a formula in X (in this case an assumption is introduced, and this is indicated with the sign ▼),

or (ii) $\varrho(\varphi)$ consists of the conclusions (possibly none) of an instance of R, whose premisses (possibly none) occur in φ,

or (iii) φ is a branch ending in a formula in Y (in this case a conclusion is reached on the branch, and this might be indicated with the sign ▲, but we will rather indicate branches with no conclusions by an empty end).

Conclusion tree proofs are usually written as growing down the page, and we adhere to this practice.

Some concrete examples will make the later comparison with the analogous premiss tree proofs easier. Let the rules be Knealean classical rules $C_{\supset}$:

$$\frac{A \qquad B}{A \land B} \qquad \frac{A \land B}{A} \qquad \frac{A \land B}{B} \qquad (\land\text{-rules}),$$

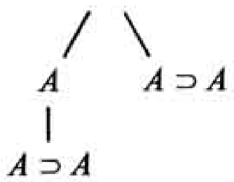
$$\frac{A}{A \lor B} \qquad \frac{B}{A \lor B} \qquad \frac{A \lor B}{A \qquad B} \qquad (\lor\text{-rules}),$$

$$\frac{B}{A \supset B} \qquad \frac{(C)}{A \qquad A \supset B} \qquad \frac{A \qquad A \supset B}{B} \qquad (\supset\text{-rules}),$$

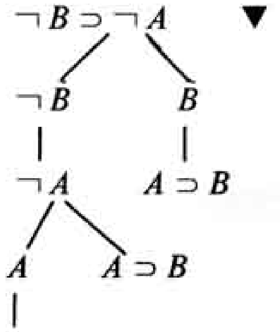
$$\frac{(C)}{A \qquad \neg A} \qquad \frac{A \qquad \neg A}{(C)} \qquad (\neg\text{-rules}).$$

Here are the examples:

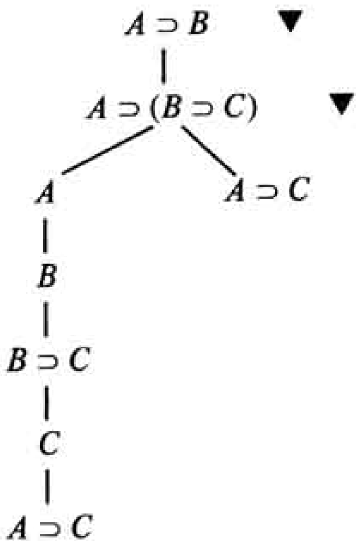
(1) $\vdash A \supset A$



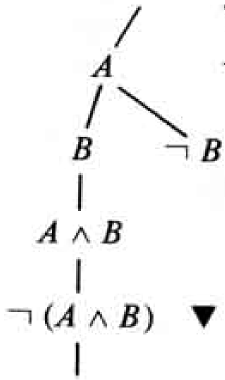
(2) $\neg B \supset \neg A \vdash A \supset B$



(3) $A \supset B,\ A \supset (B \supset C) \vdash A \supset C$



(4) $\neg(A \wedge B) \vdash \neg A, \neg B$



It is proved on [2, p. 53] that conclusion tree proofs are adequate, i.e. for any rules of inference R, conclusion-tree-proofs-deducibility by R coincides with the consequence relation characterised by R. It is important to notice, that the notion of consequence relation characterised by R is a rule theoretical notion, i.e. it does not depend on any notion of proof.

The rules $C_{\supset}$ are strictly rule theoretical. They do not pressupose any notion of proof, in contrast, for example, to the well-known ($\supset$)-rule

$$\frac{\begin{array}{c} A \quad (\blacktriangledown) \\ \vdots \\ B \end{array}}{A \supset B}$$

(A discharge of assumption is indicated by bracketing the relevant assumption sign.) The meaning of this rule depends on proofs permissible in intermediate steps (between A and B).

It is easy to prove that this proof-theoretical ($\supset$)-rule is equivalent to the two of the three Knealean rule-theoretical $\supset$-rules.

Theorem 1 (see p. 280). *Let* $C_{(\supset)}$*-rules be* $C_{\supset}$*-rules in which* $\supset$*-rules*

$$\frac{B}{A \supset B} \qquad \frac{(C)}{A \quad A \supset B}$$

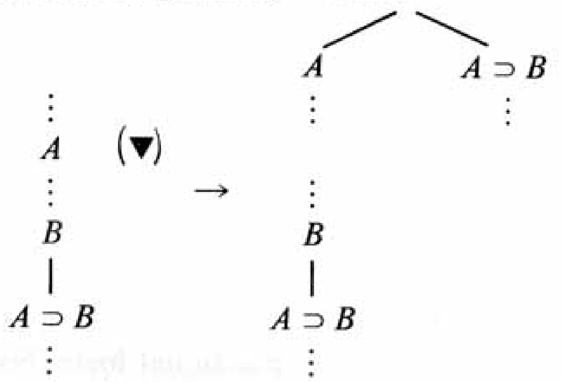
are replaced with $(\supset)$*-rule*

$$\frac{\begin{array}{c} A \quad (\blacktriangledown) \\ B \end{array}}{A \supset B}$$

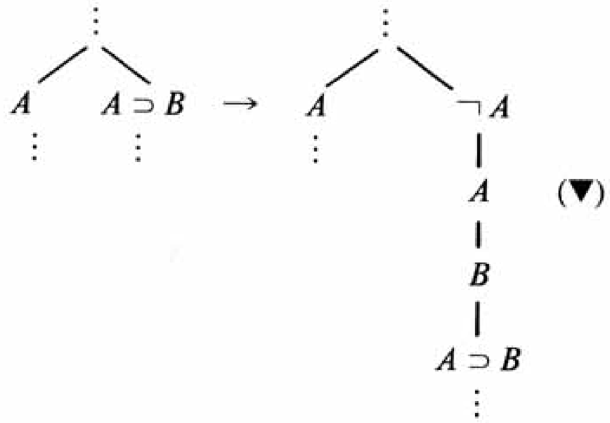
Deducibility in $\mathcal{K}(\mathrm{ctp}, \mathrm{C}_{\supset})$, *i.e. deducibility by the conclusion tree proofs and* $\mathrm{C}_{\supset}$*-rules, and deducibility in* $\mathcal{K}(\mathrm{ctp}, \mathrm{C}_{(\supset)})$, *i.e. deducibility by the conclusion tree proofs and* $\mathrm{C}_{(\supset)}$*-rules, are the same relation.*

In other words, $\mathcal{K}(\mathrm{ctp}, \mathrm{C}_{\supset})$ and $\mathcal{K}(\mathrm{ctp}, \mathrm{C}_{(\supset)})$ are two equivalent formulations of classical propositional logic.

Proof. The first scheme shows that each application of the $(\supset)$-rule is realizable by applications of the corresponding $\supset$-rules:



The second scheme shows that each application of the $\supset$-rule $\dfrac{(C)}{A \quad A \supset B}$ is realizable by an application of the $(\supset)$-rule:



It is evident that each application of the $\supset$-rule $\dfrac{B}{A \supset B}$ is realizable by a trivial application of the $(\supset)$-rule. □

The $\mathcal{K}(\mathrm{ctp}, \mathrm{C}_{(\supset)})$ formulation of classical propositional logic is especially interesting, because it is very easy to transform it into a formulation of intuitionistic propositional logic. If we eliminate $\neg$-rules, introduce falsum $\perp$, define $\neg A$ as $A \supset \perp$ and restrict the $(\supset)$-rule

$$\frac{\begin{array}{c} A \quad (\blacktriangledown) \\ B \end{array}}{A \supset B}$$

to singular applications, we get proof-theoretical conclusion-tree-proofs formulation of intuitionistic propositional logic. That is, the following theorem holds.

Theorem 2. *Deducibility in* $\mathcal{J}(\mathrm{ctp}, \mathrm{C}_{(\supset)s})$, *i.e. deducibility by the conclusion tree proofs and* $\mathrm{C}_{(\supset)s} = \mathrm{C}_{(\supset)} \setminus (\neg\text{-rules})$ *rules, such that the* $(\supset)$*-rule is restricted to singular applications, coincides with the well-known intuitionistic propositional deducibility. The singularity restriction means that the* $(\supset)$*-rule may be applied only to proofs in which all the occurrences of the endings of the branches of the proof (i.e. the consequences of the proof) are the occurrences of one and the same formula, and the rule has to be applied to all the occurrences simultaneously.*

Proof. Each of the $\mathrm{C}_{(\supset)s}$-rules, applied to a proof of the consequence $X \vdash Y$, yields a proof of the consequence $X' \vdash Y'$. The corresponding multiple sequent rules preserve intuitionistic validity of the consequences, and this can be proved in the same way it was proved in [3]. Hence each deduction in $\mathcal{J}(\mathrm{ctp}, \mathrm{C}_{(\supset)s})$ is intuitionistically valid.

On the other hand, comparing $\mathcal{J}(\mathrm{ctp}, \mathrm{C}_{(\supset)s})$ with GENTZEN's NI, it is quite easy to prove that each intuitionistically valid. Consequence is deducible in $\mathcal{J}(\mathrm{ctp}, \mathrm{C}_{(\supset)s})$. □

In the notation '$\mathrm{C}_{\supset}$' the conditional $\supset$ is emphasized, because there is no dual of $\supset$ in $\mathrm{C}_{\supset}$ (of course the dual of $\wedge$ is $\vee$, the dual of $\vee$ is $\wedge$ and the dual of $\neg$ is $\neg$). Now we introduce the missing dual $\not\subset$ with the following $\not\subset$-rules:

$$\frac{A \qquad A \not\subset B}{(C)} \qquad \frac{A \not\subset B}{B} \qquad \frac{B}{A \qquad A \not\subset B},$$

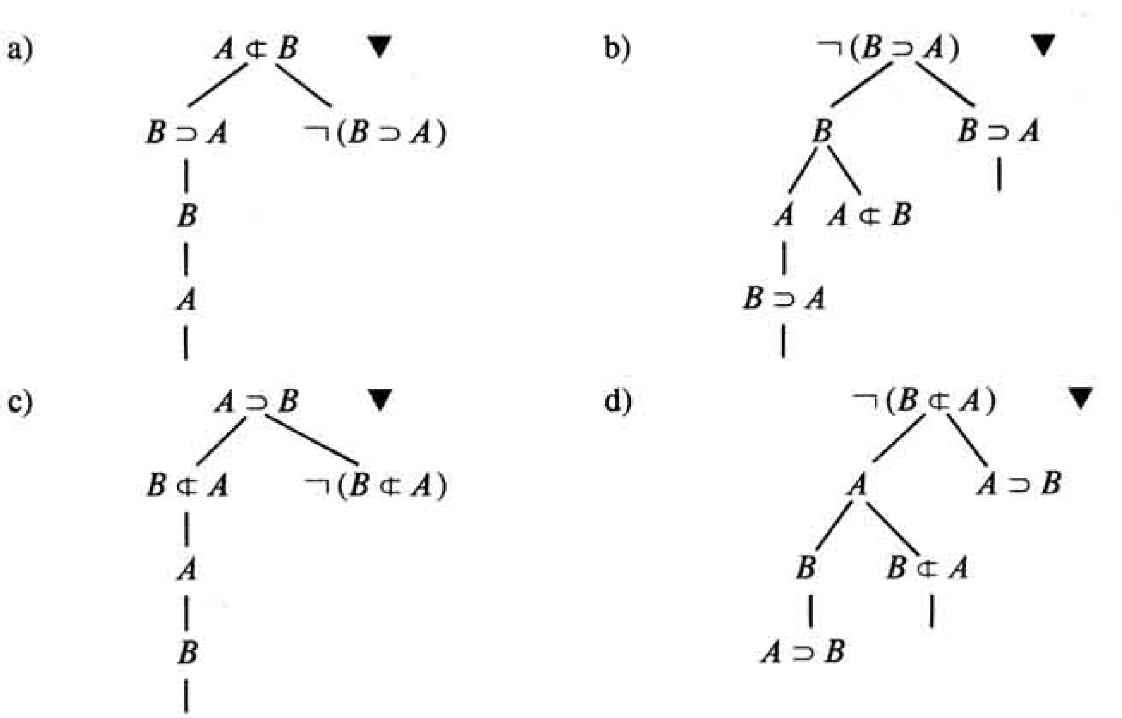
and denote the complete set of classical rules with C. Hence, $\mathrm{C} = \mathrm{C}_{\supset} \cup \not\subset$-rules. Of course instead of $\mathrm{C}_{\supset}$ or C we can also use $\mathrm{C}_{\not\subset} = \mathrm{C} \setminus \supset$-rules. This means that $\mathcal{K}(\mathrm{ctp}, \mathrm{C}_{\supset})$, $\mathcal{K}(\mathrm{ctp}, \mathrm{C}_{\not\subset})$ and $\mathcal{K}(\mathrm{ctp}, \mathrm{C})$ are three equivalent formulations of classical propositional logic. Namely, in $\mathcal{K}(\mathrm{ctp}, \mathrm{C})$ it is easy to prove that $A \not\subset B$ is equivalent to $\neg(B \supset A)$, and also that $A \supset B$ is equivalent to $\neg(B \not\subset A)$, i.e. the following theorem holds.

Theorem 3. *In* $\mathcal{K}(\mathrm{ctp}, \mathrm{C})$:

a) $A \not\subset B \vdash \neg(B \supset A)$, b) $\neg(B \supset A) \vdash A \not\subset B$,

c) $A \supset B \vdash \neg(B \not\subset A)$, d) $\neg(B \not\subset A) \vdash A \supset B$.

Proof.



□

Now, we start with our discussion of premiss tree proofs.

Definition 2. A *premiss tree proof from a set of formulae X to a set of formulae Y by rules of inferences R* is a finite tree of formulae such that for each file φ

either (i) rank $\varrho(\varphi)$ consists of an occurrence of a formula in Y (in this case a conclusion is introduced, and this is indicated with the sign ▲),

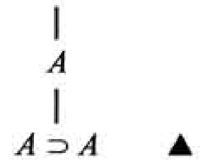
or (ii) $\varrho(\varphi)$ consists of the premisses (possibly none) of an instance of R, whose conclusions (possibly none) occur in φ,

or (iii) φ is a branch ending in a formula in X (in this case an assumption is reached on the branch, and this might be indicated with the sign ▼, but we will rather indicate branches with no assumptions by an empty end).

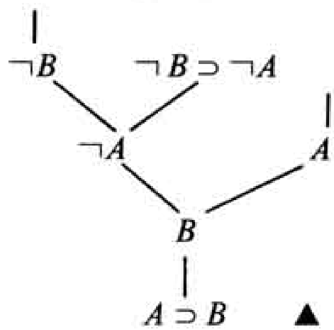
Premiss tree proofs will be written as growing up the page.

As examples, we give the premiss tree proofs (by $C_{\supset}$-rules) of the same consequences that served as examples for conclusion tree proofs.

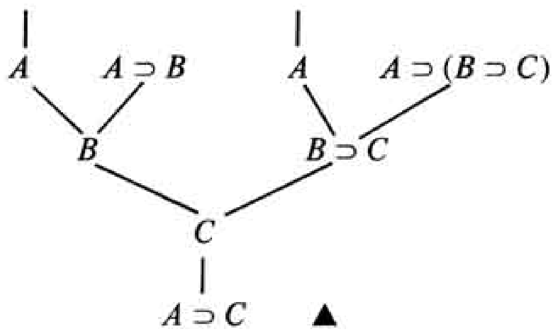
(1') $\vdash A \supset A$



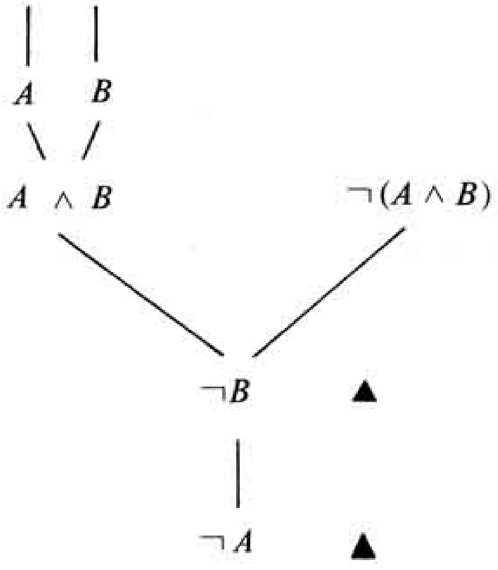
(2') $\neg B \supset \neg A \vdash A \supset B$



(3') $A \supset B,\ A \supset (B \supset C) \vdash A \supset C$



(4′) $\neg(A \wedge B) \vdash \neg A, \neg B$



Premiss tree proofs are adequate, and the proof of this fact is analogous to the already mentioned proof of the adequacy of the conclusion tree proofs (cf. [2, p. 53]).

Theorem 4. *Premiss tree proofs are adequate.*

Proof. Let R by any rules of inferences (in the sense of [2, p. 39f.]), and let $\vdash_R$ be the consequence relation characterised by R (in the sence of [2, p. 40f.]).

Suppose that $X \nvdash_R Y$. Then there exists a partition (T, U), that satisfies R, such that $X \subset T$ and $Y \subset U$. For any premiss tree proof Π with branches ending in formulae in X, let $C_1, C_2, \ldots, C_{i-1}$ be a maximal file of Π such that $C_j \in U$ for each $j \leqq i-1$. Note that this maximal file cannot be a branch, because $X \subset T$, hence there exists at least one immediate successor of C_{i-1}. The immediate successors of C_{i-1} belong to T, so they can not be the premisses of an instance of R, whose conclusions belong to the file $C_1, C_2, \ldots, C_{i-1}$. The immediate successors of C_{i-1} belong to U, so they can not belong to $X \subset T$. Therefore, Π is not a premiss tree proof from X to Y by R, for any premiss tree proof Π with branches ending in formulae in X. This means that Y is not a premiss-tree-proof deducible from X by R. Hence, it is proved that $\vdash_R$ includes premiss-tree-proof deducibility by R.

On the other hand, it is a matter of straightforward verification to prove that premiss-tree-proof deducibility by R includes all instances of R and that it is closed under overlap, dilution and cut for formulae, which, together with theorem 2.16 in [2, p. 40], proves that premiss-tree-proof deducibility by R includes $\vdash_R$. □

Corollary. $\mathcal{K}(\text{ptp}, C_\supset)$, $\mathcal{K}(\text{ptp}, C_\not\subset)$ *and* $\mathcal{K}(\text{ptp}, C)$, *i.e. deducibility by the premiss tree proofs and* $C_\supset$, $C_\not\subset$, C-*rules respectively, are three equivalent formulations of classical propositional logic.*

Proof. According to Theorem 4, $\vdash_C = \mathcal{K}(\text{ptp}, C)$ (and hence, it also follows that $A \not\subset B$ is equivalent to $\neg(B \supset A)$, and that $A \supset B$ is equivalent to $\neg(B \not\subset A)$). □

We may remind that in theorem 1 it is shown how to replace the two of the three rule-theoretical rules with the proof-theoretical ($\supset$)-rule. Now, with the help of the newly introduced premiss tree proofs it is possible to replace two of the three rule-theoretical $\not\subset$-rules with the following proof-theoretical ($\not\subset$)-rule

$$\begin{array}{l} \dfrac{A \not\subset B}{B} \\ \;\;A \qquad\quad (\blacktriangle) \end{array}$$

in which a consequence is discharged (namely, the premiss tree proofs are the appropriate kind of proofs for solving this problem of replacement, because they admit of discharging a consequence).

Theorem 5. *Let* $C_{(\not\subset)}$*-rules be* $C_{\not\subset}$*-rules in which* $\not\subset$*-rules*

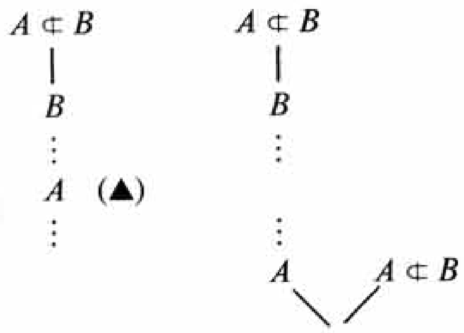
$$\frac{A \not\subset B}{B} \qquad\qquad \frac{A \qquad A \not\subset B}{(C)}$$

are replaced with the following ($\not\subset$)*-rule*:

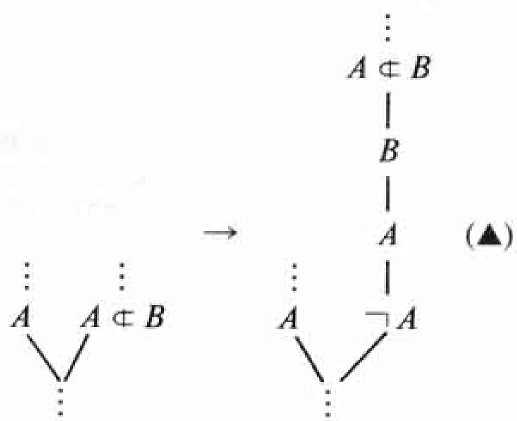
$$\begin{array}{l} \dfrac{A \not\subset B}{B} \\ \;\;A \qquad\quad (\blacktriangle). \end{array}$$

Deducibility in $\mathcal{K}(\text{ptp}, C_{\not\subset})$ *and deducibility in* $\mathcal{K}(\text{ptp}, C_{(\not\subset)})$ *are the same relation.*

Proof. The first scheme shows that each application of the ($\not\subset$)-rule is realizable by applications of the corresponding $\not\subset$-rules:



The second scheme shows that each application of $\not\subset$-rule $\dfrac{A \qquad A \not\subset B}{(C)}$ is realizable by an application of ($\not\subset$)-rule:



It is evident that each application of $\not\subset$-rule $\dfrac{A \not\subset B}{B}$ is realizable by a trivial application of the ($\not\subset$)-rule. □

In the same way as we transformed classical $\mathcal{K}(\mathrm{ctp}, C_{(\supset)})$ to intuitionistic $\mathcal{J}(\mathrm{ctp}, C_{(\supset)s})$, we transform dual-classical, i.e. classical $\mathcal{K}(\mathrm{ptp}, C_{(\not\subset)})$, to dual-intuitionistic $\mathcal{J}(\mathrm{ptp}, C_{(\not\subset)s})$.

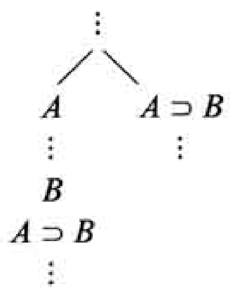
Definition 3. *Let* $C_{(\not\subset)s} = C_{(\not\subset)} \setminus (\neg\text{-}rules)$. *Deducibility in* $\mathcal{J}(\mathrm{ptp}, C_{(\not\subset)s})$ *is deducibility by the premiss tree proofs and* $C_{(\not\subset)s}$ *rules, such that the* ($\not\subset$)*-rule is restricted to singular application. The singularity restriction means that the* ($\not\subset$)*-rule may be applied only to the proofs in which all the occurences of the endings of the branches of the proof* (*i.e. the assumptions of the proof*) *are the occurences of one and the same formula, and the rule has to be applied to all the occurences simultaneously. Dual-intuitionistic deducibility is deducibility in* $\mathcal{J}(\mathrm{ptp}, C_{(\not\subset)s})$.

In the same way as we proved theorem 2 we can prove the following theorem:

Theorem 6 (cf. [1]). *Deducibility in* $\mathcal{J}(\mathrm{ptp}, C_{(\not\subset)s})$ *coincides with the deducibility in Goodman's logic of contradiction.*

As we already said, we hope that this result might make premiss tree proofs more interesting, and we think that it also might be of help for a better understanding of GOODMAN's logic, i.e. dual-intuitionistic logic.

Added in proof: After reading the paper Prof. Šeper remarked that theorem 1 (and therefore its consequence 2 and their duals 5 and 6) does not hold. Namely, if we apply premiss tree proofs to the proof-theoretical ($\supset$)-rule, it is not sufficient to discharge its assumption *A but also all the formulae on the file from A to B* (*included*). We hope it would be easy to see that this change makes theorem 1 (and all the rest) valid. Also, the right-hand side of the first scheme of theorem 1 should be as follows:



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[2] SHOESMITH, D. J., and T. J. SMILEY, Multiple-conclusion logic. Cambridge University Press, Cambridge–New York 1978.
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