SINGULAR CONSEQUENCE RELATIONS AND ORDER RELATIONS

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I In |S| we considered the multiple consequence relation as defined in |S|. The definition of the multiple consequence relation is prompted by the following line of argument: To say that a set of conclusions follows from a given set of premisses is to say that at least one of the conclusions must be true if the premisses are all true. That means that each possible state of affairs in which all the premisses are true is one in which some of the conclusions are true. Assuming the formulae of our language $\mathcal{L}$ to be capable of truth and untruth, each relevant state of affairs is represented by the partition (T,U) of formulae of $\mathcal{L}$ such that the formulae of T are true in this state of affairs, while the formulae of U are untrue in it. If $\mathcal{M}$ is the set of all partitions which correspond to the possible states of affairs, it is plausible to define the consequence relation with regard to $\mathcal{M}$ as follows.

Definition 1. Let X and Y be sets of formulae of $\mathcal{L}$ i.e. $X \subset \mathcal{L}$ and $Y \subset \mathcal{L}$. Y is a consequence of X with regard to $\mathcal{M}$ i.e.

$$X \VDash_{\mathcal{M}} Y$$

iff there is no $(T,U) \in \mathcal{M}$ such that $X \subset T$ and $Y \subset U$. It is also said that the set of partitions $\mathcal{M}$ generates the consequence relation $\VDash_{\mathcal{M}}$.

If we presuppose nothing about the internal structure of the formulae of $\mathcal{L}$, and about their semantical interconnections, we have to be prepared to allow any set of partitions of formulae to play the role of $\mathcal{M}$. So, we are led to the general definition of the multiple consequence relation proposed in |S|.

Definition 2. A relation $\Vdash$ on $\mathcal{P}(\mathcal{L})$ is a consequence relation iff there is a set of partitions $\mathcal{M}$ such that $\Vdash\ = \ \Vdash_{\mathcal{M}}$.

Remark: Each set of partitions $\mathcal{M}$ determines, and is completely determined, by $\mathcal{T} = \{T: (\exists\ U)(T,U) \in \mathcal{M}\}$ which will be called its true set, or by $\mathcal{U} = \{U: (\exists\ T)(T,U) \in \mathcal{M}\}$ which will be called its untrue set. So, we will talk about consequence with regard to $\mathcal{M}$, or with regard to $\mathcal{T}$, or with regard to $\mathcal{U}$, synonymously and we will use the notations $\Vdash_{\mathcal{M}}$, $\Vdash_{\mathcal{T}}$, $\Vdash_{\mathcal{U}}$ interchangeably.

We consider the three special cases (i.e. restrctions) of the consequence relation $\Vdash$.

(i) The most common single-conclusion relation which permits only one conclusion and which accordingly has instances of the form

$$X \Vdash B \qquad X \subset \mathcal{L}, \quad B \in \mathcal{L}.$$

(ii) The single-premiss relation which permits only one premiss and which accordingly has instances of the form

$$A \models Y \qquad A \in \mathcal{L}, \quad Y \subset \mathcal{L}.$$

(iii) The singular relation which permits only one premiss and only one conclusion, and which accordingly has instances of the form

$$A \vdash B \qquad A \in \mathcal{L}, \quad B \in \mathcal{L}.$$

The third relation is our main interest in this paper.

It is evident that these restrictions satisfy

(1) $$\vdash \; = \; \Vdash \cap \models$$

The characterization theorem, proved in |S| p. 16, states that a single-conclusion relation is a consequence relation iff it is closed under

overlap $B \in X \rightarrow X \Vdash B$
dilution $X' \Vdash B \;\&\; X' \subset X \rightarrow X \Vdash B$, and
cut for sets $X, Z \Vdash B \;\&\; (\forall A \in Z)\, X \Vdash A \rightarrow X \Vdash B$.

Results on counterparts, in ch. 5 of |S| (p. 72-74.), may be understand as proving that different sets of partitions may generate one and the same single-conclusion consequence relation.

Considering single-conclusion (single-premiss) consequence relations as closure operators, in a Tarskian way, we found in |Š| the simple conditions that must be satisfied by different sets of partitions in order to generate the same consequence relation. Incidentally the characterization theorem for single-conclusion (single-premiss) consequence relations turned out to be an immediate corollary to the characterization theorem for closure operators.

We quote the relevant definitions, lemmas, theorems and corollaries; proofs can be found in |Š|.

A single-conclusion relation $\Vdash$, with instances of the form $X \Vdash B$ ($X \subset \mathcal{L}$, $B \in \mathcal{L}$), determines and is completely determined by the corresponding consequence-operator

$$\Vdash\; : \wp(\mathcal{L}) \rightarrow \wp(\mathcal{L}) \quad \text{definied by} \quad \Vdash_{X} = \{B : X \Vdash B\}.$$

A consequence-operator $\Vdash^{\mathcal{T}}$, which corresponds to $\Vdash_{\mathcal{T}}$, may be characterized refering directly to the true set $\mathcal{T}'$. This is the content of the following lemma.

Lemma 1. A consequence-operator $\Vdash^{\mathcal{T}}$ corresponds to the single-conclusion consequence relation $\Vdash_{\mathcal{T}}$ iff
$\overset{\Vdash\mathcal{T}}{X} = \bigcap\{T : X \subset T \ \& \ T \in \mathcal{T}\}$.

Hence, the consequence-operator $\Vdash^{\mathcal{T}} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is a closure operator on $\mathcal{L}$, in the sense of the following definition.

Definition 3. Let $\mathcal{L}$ be any set and let $\mathcal{T}$ be any set of subsets of $\mathcal{L}$, i.e. $\mathcal{T} \subset \mathcal{P}(\mathcal{L})$. The closure operator generated by $\mathcal{T}$ is a function $\overline{}^{\mathcal{T}} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$, such that
$\overline{X}^{\mathcal{T}} = \bigcap\{T : X \subset T \ \& \ T \in \mathcal{T}\}$.
A function $\overline{} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is a closure operator on $\mathcal{L}$ iff there exists $\mathcal{T} \subset \mathcal{P}(\mathcal{L})$ such that $\overline{} = \overline{}^{\mathcal{T}}$.

Note that we impose no condition on $\mathcal{T}$ (such as the intersection property or something else). $\mathcal{T}$ may be any subset of $\mathcal{P}(\mathcal{L})$.

Lemma 2. Any closure operator $\overline{}$ has the following properties:

1. $X \subset \overline{X}$, 2. $Y \subset X \Rightarrow \overline{Y} \subset \overline{X}$ and 3. $\overline{X} = \overline{\overline{X}}$.

Lemma 3. A function $\overline{} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ with properties 1, 2. and 3. (from lemma 2.) is a closure operator on $\mathcal{L}$.

Definition 4. The canonical extension of $\mathcal{T} \subset \mathcal{L}$ is the set $\overline{\mathcal{T}} = \{T : T \subset \mathcal{L} \ \& \ T = \overline{T}^{\mathcal{T}}\}$.

Lemma 4.

(i) $\mathcal{T} \subset \overline{\mathcal{T}}$ and $\overline{\mathcal{T}} = \overline{\overline{\mathcal{T}}}$.

(ii) $\overline{}^{\mathcal{T}} = \overline{}^{\overline{\mathcal{T}}}$.

(iii) $\overline{}^{\mathcal{T}_1} = \overline{}^{\mathcal{T}_2}$ iff $\overline{\mathcal{T}}_1 = \overline{\mathcal{T}}_2$.

Lemma 5. $\overline{\mathcal{T}}$ is the largest generator of $\overline{}^{\mathcal{T}}$.

Lemma 6. The canonical extension of $\mathcal{T} \subset \mathcal{L}$ is the intersection extension i.e. $\overline{\mathcal{T}} = \{ X : (\exists \mathcal{A})(\mathcal{A} \subset \mathcal{T}\ \&\ X = \bigcap \mathcal{A})$.

Taking into account that a consequence-operator is a closure operator, the characterization theorem and the equivalent generators theorem follow at once.

Characterization theorem 1. An operator $^{-} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is a consequence-operator iff 1. $X \subset \overline{X}$, 2. $Y \subset X \to \overline{Y} \subset \overline{X}$ and 3. $\overline{X} = \overline{\overline{X}}$.

Equivalent generators theorem 1. True (untrue) sets generate one and the same consequence-operator iff their intersection (union) extension is the same. The extension is the largest generator of the consequence-operator.

Taking into account the simple and natural connection between consequence-operators and single-conclusion consequence relations, we derive the characterization theorem of $|S|$, and the sought for characterization of equivalent generators, as simple corrolaries.

Corrolary 1. A single-conclusion relation is a consequence relation iff it is closed under overlap, dilution and cut for sets.

Corrolary 2. True (untrue) sets generate one and the same single-conclusion consequence relation iff their intersection (union) extension is the same. The extension is the largest generator of the single-conclusion consequence relation.

A single-premiss relation $\models$, with instances of the form $A \models Y$ $(A\in\mathcal{L},\ Y\subset\mathcal{L})$, determines and is completely determined by the corresponding assumption-operator $\overset{\models}{} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ defined by

$$\overset{\models}{Y} = \{A : A \models Y\}.$$

An assumption operator $\models^{\mathcal{U}}$, which corresponds to $\models_{\mathcal{U}}$, may be characterized refering directly to the untrue set $\mathcal{U}$.

Lemma 7. An assumption-operator $\models^{\mathcal{U}}$ corresponds to the single-premiss consequence relation $\models_{\mathcal{U}}$ iff $\overset{\models^{\mathcal{U}}}{Y} = \bigcap\{U : Y\subset U \ \& \ \& \ U\in\mathcal{U}\}$.

Hence, the assumption-operator $\models^{\mathcal{U}} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is a closure operator on $\mathcal{L}$, and the characterization theorem, as well as the equivalent generators theorem, are as before.

Characterization theorem 2. An operator $\overline{} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is an assumption-operator iff 1. $X\subset\overline{X}$, 2. $Y\subset X \to \overline{Y}\subset\overline{X}$ and 3. $\overline{X} = \overline{\overline{X}}$.

Equivalent generators theorem 2. Untrue (true) sets generate one and the same assumption-operator iff their intersection (union) extension is the same. The extension is the greatest generator of the assumption-operator.

Taking into account the simple and natural connection between assumption-operators and single-premiss consequence relations we easily derive the corrolaries:

Corrolary 3. A single-premiss relation is a consequence relation iff it is closed under

overlap $A \in Y \rightarrow A \models Y$,

dilution $A \models Y' \;\&\; Y' \subset Y \rightarrow A \models Y$ and

cut for sets $A \models Y, Z \;\&\; (\forall B \in Z) B \models Y \rightarrow A \models Y$.

Corrolary 4. Untrue (true) sets generate one and the same single-premiss consequence relation iff their intersection (union) extension is the same. The extension is the largest generator of the single-premiss consequence relation.

Let $\mathcal{M}$ be a set of partitions of $\mathcal{L}$, $\mathcal{T} = \{T : (\exists U)(T,U) \in \mathcal{M}\}$ its true set and $\mathcal{U} = \{U : (\exists T)(T,U) \in \mathcal{M}\}$ its untrue set. From

(1) follows

(2) $$\vdash_{\mathcal{M}} = \Vdash_{\mathcal{T}} \cap \models_{\mathcal{U}}.$$

Taking into account (2), corrolaries 1. and 3. imply

Corrolary 5. A singular relation $\vdash$ is a consequence relation iff it is closed under

overlap $A \vdash A$ and

dilution $A \vdash B \;\&\; B \vdash C \rightarrow A \vdash C$.

Corrolary 6. is a simple consequence of corrolaries 2. and 4.

Corrolary 6. True (untrue) sets generate one and the same singular consequence relation iff their intersection-union extension is the same. The extension is the largest generator of the singular consequence relation (i.e. the largest generator, true and untrue, is closed under unions and intersections).

Corollaries 5. and 6. may be seen as abstract algebraic results on preordered sets and we consider some consequences of this fact.

Definition 5. A set S is preordered by a binary relation $\trianglelefteq$ if this relation is reflexive and transitive on S:

$$\forall a \in S(a \trianglelefteq a)$$

$$\forall a,b,c \in S(a \trianglelefteq b \ \& \ b \trianglelefteq c \rightarrow a \trianglelefteq c).$$

Corollary 5'. Every family $\mathcal{T}$ of the subsets of S (i.e. $\mathcal{T} \subset \mathcal{P}(S)$) generates the unique preorder on S defined with

(3) $a \trianglelefteq b$ iff $-(\exists T \in \mathcal{T})(a \notin T \ \& \ b \in T)$

and conversely every preorder on S is generated by a family of the subsets of S.

Corollary 6'. Two families of the subsets of S generate the same preorder on S iff they have the same intersection-union extension and this extension is the largest family that generates this preorder.

Every (partial) order on S is a preorder on S; namely, it is the preorder which is antisymmetric on S:

$$\forall a,b \in S \ (a \trianglelefteq b \ \& \ b \trianglelefteq a \rightarrow a = b).$$

On the other hand to every preorder on S there corespond the unique order $\leqslant \ = \ \trianglelefteq/\sim$ on $S/\sim$ where $\sim$ is the relation of equivalence on S defined as follows:

(4) $a \sim b$ iff $(a \trianglelefteq b \ \& \ b \trianglelefteq a)$.

Hence, the following corollary is true.

Corollary 7. Each family $\mathcal{T}$ of the subsets of S (i.e.

$\mathcal{T}\subset\mathcal{P}(S)$) generates the unique order $\leq \,=\, \trianglelefteq/\!\sim$ on $S/\!\sim$ defined by (3) and (4).

The order generated by a family $\mathcal{T}$ of the subsets of S is an order on S iff $\sim$ coincides with $=$. This happens if and only if

(5) $$(\forall a,b\in S)(\exists T\in\mathcal{T})(a\notin T \;\&\; b\in T)\vee(a\in T \;\&\; b\notin T),$$

that is iff each pair of the elements of S can be separated with an element of $\mathcal{T}$. Hence, the following corollary is true.

Corollary 8. Every family $\mathcal{T}$ of the subsets of S, which satisfy (5) generates the unique order $\leq$ on S defined with

(6) $$a\leq b \quad \text{iff} \quad \neg(\exists T\in\mathcal{T})(a\notin T \;\&\; b\in T).$$

We can also prove the converse of this corollary:

Corollary 9. Every order on S is generated by a family of the subsets of S which satisfies (5). Moreover, the largest family that generates the order (i.e. the intersection-union extension of any family which generates it) is the family of the initial segments of the order.

Proof. It is easy to check that the family of the initial segments of order $\leq$ generates the order. Now, let $\mathcal{T}$ be the largest family that generates $\leq$ and let $T\in\mathcal{T}$. We have to prove that T is an initial segment of $\leq$. That is, we have to prove:

$$(b\in T \;\&\; a\leq b)\to a\in T \qquad \text{i.e.}$$
$$(b\in T \;\&\; a\notin T)\to a\not\leq b \qquad \text{i.e.}$$
$$(b\in T \;\&\; a\notin T)\to(\exists T\in\mathcal{T})(b\in T \;\&\; a\notin T)$$

which is certainly true.

Now we know that an order is a total order iff the set of the initial segments of the order satisfies

$$(7) \qquad \forall T_1, T_2 \in \mathcal{T} \ (T_1 \subset T_2 \vee T_2 \subset T_1)$$

Hence the following corollary is true.

Corollary 10. Every family $\mathcal{T}$ of the subsets of S, which satisfies (5) and (7), generates the unique total order $\leq$ on S defined with (6) and conversely, every total order on S is generated by a family of the subsets of S which satisfies (5) and (7). Moreover, the largest family that generates the order (i.e. the intersection-union extension of any family which generates it) is the family of the initial segments of the order.

Our final remark is that corollaries 5'-10 show that we could define a preorder on S to be a subset of $\mathcal{P}(S)$, an order on S to be a subset $\mathcal{T}$ of $\mathcal{P}(S)$ such that

$$(5) \qquad (\forall a,b \in S)(\exists T \in \mathcal{T})(a \notin T \ \& \ b \in T) \vee (a \in T \ \& \ b \notin T)$$

and a total order on S to be a subset $\mathcal{T}$ of $\mathcal{P}(S)$ such that (5) and

$$(7) \qquad (\forall T_1, T_2 \in \mathcal{T})(T_1 \subset T_2 \vee T_2 \subset T_1).$$

|S| **Shoesmith, D.J., Smiley, T.J.** Multiple-conclusion logic, Cambridge, 1978.

|Š| Šikić, Z. Closure operators and consequence relations, to appear.

Zwangläufe im höchstens vierdimensionalen euklidischen Raum $\mathbb{E}_\nu$ mit affin durchlaufenen Bahnpunktsmengen

H. Florian in kollegialer Verbundenheit
zum 65. Geburtstag gewidmet.

H. Vogler