INT. J. MATH. EDUC. SCI. TECHNOL., 1990, VOL. 21, NO. 1, 111–115

Taylor's theorem

by ZVONIMIR ŠIKIĆ

Faculty of Mechanical Engineering and Naval Architecture,
University of Zagreb, 41000 Zagreb, Yugoslavia

(Received 28 October 1987; revision received 11 April 1988)

Two problems have to be considered when introducing Taylor's formula into a calculus course: motivation for the use of the Taylor polynomial as an approximate function and the choice among different proofs of Taylor's theorem. A number of solutions found in the literature are discussed.

Concerning the first problem, we think that the best solution is to find a proof of Taylor's theorem which generates the Taylor polynomial, so that we can dispense with any independent motivation for the polynomial. Such a proof is given at the end of the paper. Concerning the second problem, it is shown that the most common type of proof of Taylor's theorem presents a significant psychological difficulty. A proof which avoids the difficulty is presented, but I nevertheless think that the proof at the end of the paper is still the best choice.

Introducing Taylor's formula into a calculus course implies considering two problems:

(i) motivation for the use of the Taylor polynomial as an approximate function;
(ii) choosing from the different proofs of Taylor's theorem.

One 'solution' to problem (i) is not to motivate the polynomial at all (see, for example, [1–3]). It is not accidental that such great mathematicians as Banach, Hardy and Kuratowski single out this 'solution'. Probably the most common motivation is to prove that each polynomial is its own Taylor polynomial [4–7]. Students may rightly complain that this is a motivation to approximate a polynomial by its Taylor polynomial, which is trivial. A much better motivation is to prove that each power series (i.e. each function representable in this way) is the limit of the sequence of its Taylor polynomials [8, 9]. The shortcoming of this motivation is that it postpones the introduction of Taylor's formula (compare [8] and [9] with other references). Another motivation is to prove that a function and its Taylor polynomial have the same derivatives in their common point (see [10]). But, without Taylor's formula, it is difficult to show that the coincidence of derivatives implies a good approximation (see [11]). A very good motivation is to prove that polynomials of degree n, which coincide with a function at n different points, converge to its Taylor polynomial when all the points converge to the same point ([4] gives an indication of such an approach). This numerically oriented solution of problem (i) is time-consuming and therefore not always achievable in practice. Finally, we could dispense with problem (i). It is possible that the proof of Taylor's theorem does not need a justification of the Taylor polynomial; perhaps it may even generate it. We will shortly present such a proof.

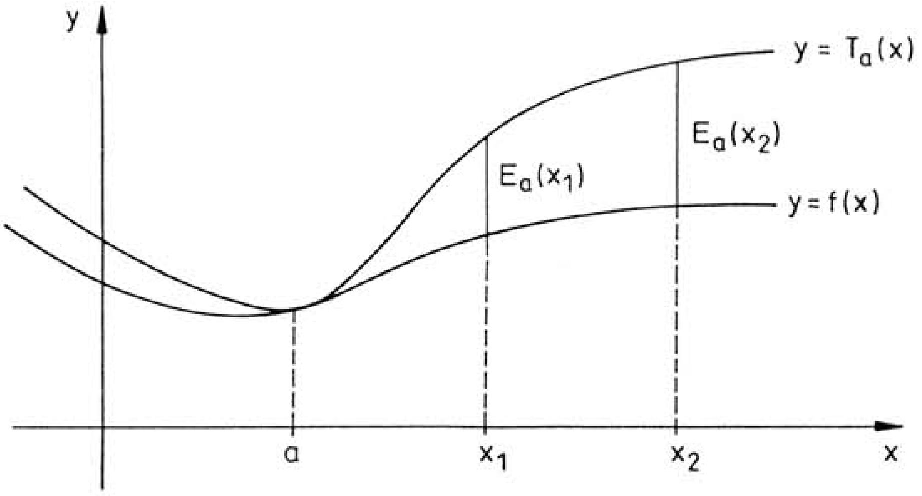

Figure 1.

Let us consider problem (ii). To prove Taylor's theorem means estimating the error $E_a(x)$, introduced by approximating a function $f(x)$ by its Taylor polynomial $T_a(x)$ (in all proofs, the function f and the degree n are kept constant).

$$T_a(x)=f(a)+f'(a)\frac{(x-a)}{1!}+f''(a)\frac{(x-a)^2}{2!}+\ldots+f^{(n)}(a)\frac{(x-a)^n}{n!} \tag{1}$$

$$E_a(x)=f(x)-T_a(x) \tag{2}$$

Accordingly, a picture which students associate with proof of Taylor's theorem is shown in figure 1. Unfortunately, in the most common type of proof, the error $E_a(x)$ is calculated with constant x and variable a (some of the proofs are based on Cauchy's mean value theorem [3, 5], and others on Rolle's theorem [1, 2, 7, 8], but this is a minor difference). The unknown function $E_a(x)$ is found as a function of variable a (it is derived with respect to a in the course of the proof), and with x as a non-varying parameter. Students to whom this subject is introduced with the motivation for Taylor polynomial, comprehend $E_a(x)$ as the difference between $f(x)$ and Taylor polynomial $T_a(x)$, implying that their visual conception is that of figure 1. It is difficult for them to understand the proof because the picture which they should associate with the proof is as shown in figure 2. This seems to be the reason for not justifying the polynomial $T_a(x)$ in [1–3]. The great mathematicians understood what problems the motivation for Taylor polynomial causes for their proofs.

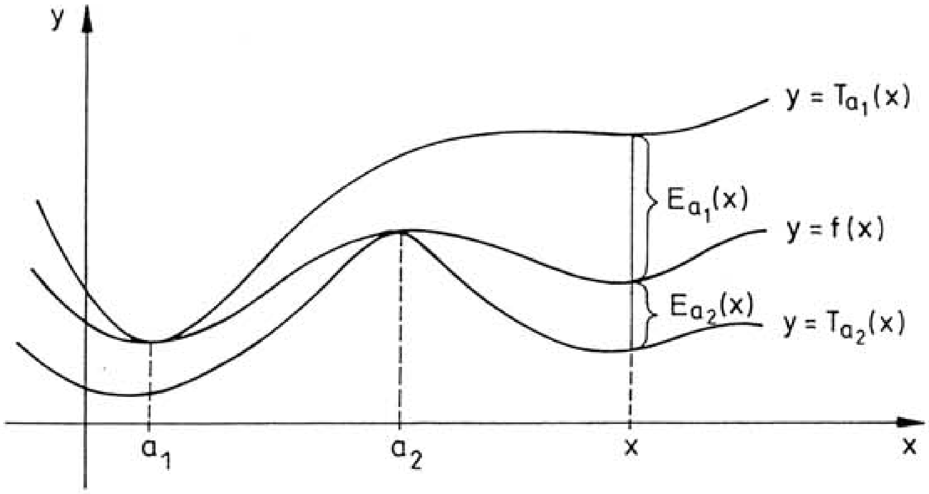

Figure 2.

Nevertheless, there is a simple proof which treats $E_a(x)$ as a function of x, with parameter a. This proof is in accordance with the visual conception of figure 1, which is proposed to our students. Here is the proof.

It is easy to check that

$$T_a^{(k)}(a) = f^{(k)}(a) \qquad \text{for } k \leqslant n \tag{3}$$

Moreover, if $T_a(x)$ has previously been justified, this basic property must already have been noticed. From (1), (2) and (3) we obtain

$$E_a^{(k)}(a) = 0 \qquad \text{for } k \leqslant n \tag{4}$$

$$E_a^{(n+1)}(x) = f^{(n+1)}(x) \tag{5}$$

According to Cauchy's mean-value theorem (C), we obtain

$$\frac{E_a(x)}{(x-a)^{(n+1)}} \overset{(4)}{=} \frac{E_a(x)-E_a(a)}{(x-a)^{n+1}-(a-a)^{n+1}} \overset{(C)}{=} \frac{E'_a(x_1)}{(n+1)(x_1-a)^n}$$

$$\overset{(4)}{=} \frac{E'_a(x_1)-E'_a(a)}{(n+1)(x_1-a)^n-(n+1)(a-a)^n} \overset{(C)}{=} \frac{E''_a(x_2)}{(n+1)n(x_2-a)^{n-1}}$$

$$\overset{(4)}{=} \frac{E''_a(x_2)-E''_a(a)}{(n+1)n(x_2-a)^{n-1}-(n+1)n(a-a)^{n-1}} \overset{(C)}{=} \cdots \overset{(C)}{=} \frac{E_a^{(n+1)}(x_{n+1})}{(n+1)!}$$

$$\overset{(5)}{=} \frac{1}{(n+1)!} f^{(n+1)}(x_{n+1}) \qquad x_{n+1} = a+\theta x, \quad 0 \leqslant \theta \leqslant 1$$

This implies

$$E_a(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a+\theta x) \qquad 0 \leqslant \theta \leqslant 1$$

Such a proof cannot be found in any of the textbooks analysed. We suggest that this approach provides a way of treating Taylor's theorem clearly and comprehensibly for students of differential calculus.

Proof's considered until now do not involve applications of integral calculus. If we admit these methods (i.e. postpone the introduction of Taylor's formula) we can dispense with problem (i). For example, a proof based on repeated integration by parts generates a Taylor polynomial [4, 9, 10, 12]. Hence, there is no need of prior justification. But students may rightly ask how the idea of repeated integration by parts occurs to anyone. This could reinforce their belief that mathematical proofs are arbitrary creations. However, there is a proof of Taylor's theorem which is not an arbitrary creation.

We teach our students of science and engineering to solve their problems by applying the following basic method of calculus:

If your problem is to find an unknown function f (whose value $f(a)=0$ is known), find the rate of change of f and integrate it from a to x.

We apply the same method in the proof of Taylor's theorem. To estimate the error of approximating the value $f(a)$ with the value $f(x)$ we have to find an unknown error function E_0 so that

$$f(a) = f(x) + E_0(x) \tag{6}$$

It is easy to see that $E_0(a)=0$, so we will apply our method. If we differentiate (6) we shall find $E'_0(x)$:

$$0=f'(x)+E'_0(x)$$

i.e.

$$E'_0(x)=-f'(x)$$

If we integrate the above we obtain

$$E_0(x)=-\int_a^x f'(x)\,dx$$

Via the mean value theorem

$$E_0(x)=-f'(\xi)\int_a^x dx=f'(\xi)(a-x)$$

Therefore by substituting the above in (6) we obtain

$$f(a)=f(x)+F'(\xi)(a-x) \tag{T0}$$

Further, it is possible to estimate the error that we make in substituting $f(x)$ for $f'(\xi)$ in (T0). Thus, the new problem is to find E_1 so that

$$f(a)=f(x)+f'(x)(a-x)+E_1(x) \tag{7}$$

Of course $E_1(a)=0$, and by differentiating (7) we obtain

$$0=f''(x)(a-x)+E'_1(x)$$

i.e.

$$E'_1(x)=-f''(x)(a-x)$$

If we integrate the above we obtain

$$E_1(x)-\int_a^x f''(x)(a-x)\,dx$$

Hence,

$$E_1(x)=-f''(\xi)\int_a^x (a-x)\,dx=f''(\xi)\frac{(a-x)^2}{2}$$

By substituting the above in (7) we obtain

$$f(a)=f(x)+f'(x)(a-x)+f''(\xi)\frac{(a-x)^2}{2} \tag{T1}$$

In repeating this process we obtain

$$f(a)=f(x)+f'(x)(a-x)+\ldots+f^{n-1}(x)\frac{(a-x)^{n-1}}{(n-1)!}+f^n(\xi)\frac{(a-x)^n}{n!} \tag{Tn}$$

We suggest that this approach provides a way of treating Taylor's theorem clearly and comprehensibly for students of differential and integral calculus.

References

[1] BANACH, S., 1952, *Rachunek Roziczkowy i Calkowy* (Warszawa: PWN), Chap. VI, §6.7.

[2] HARDY, G. H., 1960, *A Course of Pure Mathematics* (Cambridge University Press), p. 285.

[3] KURATOWSKY, K., 1969, *Introduction to Calculus* (Warszaw: PWN & Pergamon Press), p. 185.

[4] COURANT, R., 1965, *Introduction to Mathematical Analysis* I (New York: Wiley), p. 445.

[5] FIHTENGOLJC, G. M., 1962, *Kurs diferencijal'nogo i integral'nogo isčislenia* (Moscow: Fizmatgiz), p. 246.

[6] MYŠKIS, A. D., 1972, *Introductory Mathematics for Engineers* (Moscow: MIR Publishers), p. 161.

[7] NIKOLSKY, S. M., 1977, *A Course of Mathematical Analysis* I (Moscow: MIR Publishers), p. 151.

[8] BERMANT, A. F., and ARAMANOVICH, I. G., 1975, *Mathematical Analysis* (Moscow: MIR Publishers), p. 691.

[9] KUREPA, S., 1971, *Matematička analiza* II (Zagreb: Tehnička kujigou), p. 100.

[10] THOMAS, G. B., 1951, *Calculus and Analytic Geometry* (Reading, MA: Addison-Wesley), p. 603.

[11] ŠIKIĆ, Z., 1984, Taylorova formula i dodiri višeg reda, *Matematika*, **3,** 5–10.

[12] VAN DER WAERDEN, B. L., 1975, *Mathematik für Naturwissenschaftler* (Mannheim: Bibliographisches Institut Mannheim), p. 255.