What are numbers?

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Abstract *A number is the number of a class which is an objective, nonactual, mathematical object. The concept of class is analyzed and it is concluded that a number is the number of a pure founded class. A tempting strategy of explaining numbers away is rejected. Some well-known definitions of numbers are analyzed and it is concluded that this analysis purports the thesis that the unique notion of number does not exist. Numbers are conventional. Nevertheless, an argument is offered purporting the thesis that von Neumann's ordinal numbers are* the *ordinal numbers. Accordingly, the corresponding von Neumann's cardinal numbers are* the *numbers.*

A number is the number of something. Hence, we could start answering our question, as Frege did, by asking another one. What is a number the number of?

> But when we make a statement of number, what is that of which we assert something? This question remained unanswered[1]. (Frege, 1978, p. 58e)

Frege's answer is as straight and clear as it can be. A number is the number of a concept.

> While looking at one and the same external phenomenon, I can say with equal truth both "It is a copse" and "It is five trees", or both "Here are four companies" and "Here are 500 men". Now what changes here from one judgement to the other is neither any individual object, nor the whole, the agglomeration of them, but rather my terminology. But that is itself only a sign that one concept has been substituted for another. This suggests as the answer to the first of the questions left open in our last paragraph, that the content of a statement of number is an assertion about a concept. This is perhaps clearest with the number 0. If I say "Venus has 0 moons", there simply does not exist any moon or agglomeration of moons for anything to be asserted of; but what happens is that a property is assigned to the concept "moon of Venus", namely that of including nothing under it. If I say "the King's carriage is drawn by four horses", then I assign the number four to the concept "horse that draws the King's carriage"[2]. (Frege, 1978, p. 59e)

A concept is not something subjective like an idea. It is as objective as any object is. We assert something of a concept as truly or as falsely as we assert something of any object.

> That a statement of number should express something factual independent of our way of regarding things can surprise only those who think concept is something subjective like an idea. But this is a mistaken view. If, for example,

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> we bring the concept of body under that of what has weight, or the concept of whale under that of mammal, we are asserting something objective; but if the concepts themselves were subjective, then the subordination of one to the other, being a relation between them, would be subjective too, just as a relation between ideas is[3]. (Frege, 1978, p. 60e)

The objectivity of concepts does not imply their actuality. No more than the objectivity of the axis of the earth implies its actuality.

> I distinguish what I call objective from what is handleable or spatial or actual. The axis of the earth is objective, so is the centre of mass of the solar system, but I should not call them actual in the way the earth itself is so[4]. (Frege, 1978, p. 35e)

If something is objective but not actual it is called an abstract object. Hence, a concept is an abstract object.[5] A number is the number of an abstract object.

Now, what counts in ascribing a number to a concept is the extension of the concept. We could say, a step further from Frege,[6] that a number is the number of the *extension* of a concept. A number is the number of a class.

Classes are abstract objects too. Whatever Frege asserts about objectivity and nonactuality of concepts could be asserted, equally true, about classes. But what are classes and what classes are there? It is possible to answer this question in a first-order theory, as follows (cf. Šikić, 1994).

The membership relation $\in$ is the only undefined relation of the theory and the empty set $\varnothing$ is its only undefined object. The objects of the theory are individuals, sets and proper classes. We use capitals A, B, C, ... for any of them. Elements (individuals or sets) are defined as objects which are members of something:

$$\text{el } A \leftrightarrow \exists B(A \in B)$$

If different from $\varnothing$, classes (sets or proper classes) are defined as objects which have members:

$$\text{cl } A \leftrightarrow \exists B(B \in A) \vee A = \varnothing$$

Individuals are defined as non-classes:

$$\text{ind } A \leftrightarrow -\text{cl } A$$

Proper classes are defined as non-elements:

$$\text{pcl } A \leftrightarrow -\text{el } A$$

Sets are defined as classes which are elements, i.e. as classes small enough to be the elements of other classes:

$$\text{set} = A \leftrightarrow \text{el } A \ \& \ \text{cl } A$$

To make our formulae shorter, we use small letters x, y, z, ... for elements and capitals X, Y, Z, ... for classes:

$\leftarrow$ el: $x, y, z, \ldots \rightarrow$
$\leftarrow$ ind $\rightarrow \leftarrow$ set $\rightarrow \leftarrow$ pcl $\rightarrow$
          $\leftarrow$ cl: $X, Y, Z, \ldots \rightarrow$

That classes are extensions of concepts is expressed by the *principle of comprehension* (PC)

$$\exists Y \forall x(x \in Y \leftrightarrow \phi(x))$$

i.e. for every concept ϕ there exists a class Y such that an element x is a member of Y iff x falls under ϕ. Formula $\phi(x)$, representing a concept, may be any formula of the theory, not containing Y.

The criterion of class identity is expressed by the *principle of extensionality* (PE)

$$\forall x(x \in Y \leftrightarrow x \in Z) \leftrightarrow y = Z$$

i.e. a class Y is identical to a class Z iff Y and Z have the same members.

These two principles guarantee that for every concept ϕ there exists the unique class $\{x: \phi(x)\}$, consisting of all elements x such that x falls under ϕ. Hence, there exist the following classes:

$$\begin{aligned}
&\text{Ind} &&= \{x: \text{ind } x\}\\
&\text{V[Ind]} &&= \{x: \text{set } x\}\\
&\{x, y\} &&= \{z: z = x \vee z = y\}\\
&\text{Fin} &&= \cap\{z: \varnothing \in z \,\&\, \forall x \forall y(x \in z \,\&\, y \in z \rightarrow x \cup \{y\} \in z\}\\
& &&\quad \text{where } \cap X = \{z: \forall y(y \in X \rightarrow z \in y)\}\\
&\cup X &&= \{z: \exists y(z \in y \,\&\, y \in X)\}\\
&\text{P}X &&= \{z: \forall y(y \in z \rightarrow y \in X)
\end{aligned}$$

Ind is the class of all individuals. $\vee$ [Ind] is the universe of all sets constructed over Ind. Using Russell's argument of all sets not belonging to themselves, it is easy to prove that V[Ind] is a proper class. On the other hand, each element of V[Ind] is a set, i.e. it is small enough to be an element. $\{x, y\}$ is the (unordered) pair of x and y. Fin is the class of all finite classes. $\cap X$ is the intersection of all elements of X. $\cup X$ is the union of the elements of X. PX is the class of all subsets of X.

The next axioms assert setness, i.e. smallness, of some fundamental classes:

(A0) *Empty class* $\quad \varnothing \in \text{V[Ind]}$

i.e. the empty class is a small class.

(A2) *Pair* $\quad \forall x \forall y(\{x, y\} \in \text{V[Ind]})$

i e. the pair of small classes is a small class.

(A∞) *Infinity* $\quad \text{Fin} \in \text{V[Ind]}$

i e. the infinite class of all finite classes is a small class.

(A$\cup$) *Union* $\quad \forall x(\cup x \in \text{V[Ind]})$

i.e. the union of a small class of small classes is a small class.

(AP) *Power set* $\quad \forall x(\text{P}x \in \text{V[Ind]})$

i.e. the class of all subsets of a small class is a small class.

The axiom of infinity demands that the universe of all sets should be greater than V_ω, but it does not demand that it should be greater then $V_{\beta \cdot 2}$. (For the meaning of V_α, cf. below.) The axiom of replacement meets this demand:

(AR) *Replacement* func X & dom $X \in \mathsf{V}[\text{Ind}] \rightarrow \text{ran } X \in \mathsf{V}[\text{Ind}]$

i.e. the result of replacing the members of a small class with other small classes is a small class.

From (PE), (PC), (A0), (A2), (A_∞), ($A\cup$), (AP) and (AR), we can develop the standard theory of ordinal numbers, so as to define the proper class of all ordinals, O = {x: x is an ordinal], and to prove the principle of transfinite recursion on ordinals. According to the principle there is the unique function from O to V[Ind], with $\alpha \rightarrow \mathsf{V}_\alpha[\text{Ind}]$, such that:

$$\mathsf{V}_\varnothing[\text{Ind}] = \varnothing \quad \text{and} \quad \mathsf{V}_\alpha[\text{Ind}] = P((\bigcup_{\beta<\alpha} \mathsf{V}_\beta[\text{Ind}]) \cup \text{Ind})$$

Hence, it is possible to define the universe of all founded sets:

$$\mathsf{V}_\mathrm{O}[\text{Ind}] = \bigcup\{\mathsf{V}_\alpha[\text{Ind}]: \alpha \in \mathrm{O}\}$$

According to the definition, V_O[Ind] has the structure shown in Figure 1.

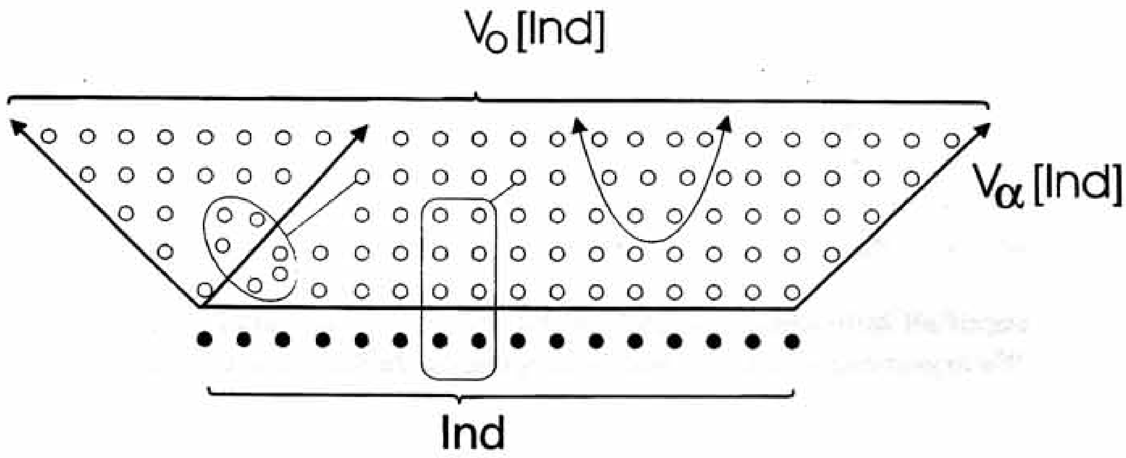

Figure 1. *The structure of* V_o *[Ind].*

At the bottom of Figure 1 there are individuals which belong neither to V_O[Ind] nor to the whole universe of sets V[Ind]. All sets whose elements are individuals or belong to the levels below a certain level α constitute V_α[Ind]. They are represented by points at level α. Hence, every point at level α represents a closed domain of points below this level. The proper classes are classes which penetrate through all levels. They are represented by open domains and there are no points to represent them at any level α.

The hierarchy of V_α[Ind] is cumulative, $\alpha \leq \beta \rightarrow \mathsf{V}_\alpha[\text{Ind}] \subseteq \mathsf{V}_\beta[\text{Ind}]$, and the union of all V_α[Ind] is V_O[Ind], the universe of all founded sets. If we start with no individuals, we get the universe of pure founded sets $\mathsf{V}_\mathrm{O} = \bigcup\{\mathsf{V}_\alpha: \alpha \in \mathrm{O}\}$, where $\mathsf{V}_\alpha = P(\bigcup_{\beta<\alpha} \mathsf{V}\beta$ and $\mathsf{V}_\varnothing = \varnothing$. The pure founded universe V_O (which is founded on no individuals, Ind = $\varnothing$) is a part of any universe V_O[Ind]. The pure part is represented by the cone on the left.

The axiom of foundation demands that there are no other sets but founded sets.

(AF) *Foundation* $\forall X \exists y(y \in X \;\&\; y \cap X = \varnothing)$

Namely, $\mathsf{V}_\mathrm{O}[\text{Ind}] = \mathsf{V}[\text{Ind}]$ is equivalent to the axiom of foundation.

Such is Cantor–Zermelo–Skolem–von Neumann–Gödel–Berneys or, in one phrase, mathematical conception of class.[7] Its first-order theory is axiomatized with (PC), (PE), (A0), (A2), (A∞), (A$\cup$), (AP), (AR), the axiom of choice (AC) and (AF).

Aczel's even broader concept admits unfounded classes (Aczel, 1988; Šikić, 1994). As far as numbers are concerned, this generalization is irrelevant. For every unfounded class there corresponds a founded one of the same size.[8] Hence, there are no new numbers which could appear in the unfounded universe.

An even further reduction is possible, namely, for every founded class there corresponds a pure one of the same size. As far as numbers are concerned and, as far as mathematics is concerned, individuals are completely unnecessary. Mathematics does not need them. It is founded on the empty set of individuals.[9] Numbers are the numbers of pure founded classes.

Now that we have learned what the objects are, of which numbers are the numbers of, we may get back to the main question. What are numbers? A tempting strategy is to explain numbers away. The meaningfulness of a statement containing a singular term does not necessarily presuppose an object named by it. We could say that:

(0) "The number 0 belongs to the class A."

(1) "The number 1 belongs to the class B."

(2) "The number 2 belongs to the class C.", etc.

is just a manner of speaking. The real meaning of the statements is given by:

(d0) "There is no member of the class A."

(d1) "There is a member y of the class B and such that the number 0 belongs to the class $B\backslash\{y\}$."

Or taking (d0) into account:

"There is a member y of the class B and such that there is no member of the class $B\backslash\{y\}$."

(d2) "There is a member z of the class C and such that the number 1 belongs to the class $C\backslash\{z\}$."

or taking (d1) into account:

"There is a member z of the class C and such that there is a member y of the class $C\backslash\{z\}$ such that there is no member of the class $C\backslash\{z\}\backslash\{y\}$.", etc.

Perhaps it is easier to grasp the meaning of these statements, if it is given in a more formal way:

(d0) $-\exists x(x \in A)$
(d1) $\exists y(y \in B \,\&\, -\exists x(x \in B\backslash\{y\}))$
(d2) $\exists z(z \in C \,\&\, \exists y(y \in C\backslash\{z\} \,\&\, -\exists x(x \in C\backslash\{z\}\backslash\{y\})))$, etc.

($x \in b\backslash\{y\}$ means $x \in B$ and $x \neq y$; $x \in C\backslash\{z\}\backslash\{y\}$ means $x \in C$, $x \neq y$ and $x \neq z$, etc.)

The problem with this strategy is that numbers have to be explained away from all contexts in which they may occur. For example, they have to be explained away from the following statements:

(i) $2 + 3 = 5$;

(ii) $\chi_1 = 2^{\chi_0}$;
(iii) Julius Caesar $\notin$ N (i.e. Julius Caesar is not a natural number);
(iv) $\forall x \forall y \forall z \forall n (x \in \mathrm{N} \ \& \ y \in \mathrm{N} \ \& \ z \in \mathrm{N} \ \& \ n \in \mathrm{N} \ \& \ n > 2 \rightarrow x^n + y^n \neq z^n)$.

It is fairly easy done in examples (i) or (ii):

(di) If X and Y are any two disjoint classes, and if the number 2 belongs to the class X and the number 3 belongs to the class Y, then the number 5 belongs to their union $X \cup Y$.

Of course, the real meaning of the expressions "the number 2 belongs to the class X", "the number 3 belongs to the class Y" and "the number 5 belongs to the class $X \cup Y$" is given by (d2), (d3) and (d5).

(dii) Every infinite subclass of PFin is in one-to-one correspondence with Fin or PFin.

The definitions of the class Fin and the power set operator P were given earlier.

Following Frege's techniques (but not his spirit) the numbers could also be explained away from (iii) or (iv). (Admittedly, not as easily as they were explained away from (i) or (ii) and, perhaps, not to everyone's satisfaction.) Nevertheless, there are many other mathematical contexts from which it is quite impossible to explain numbers away. The most common contexts are those which mention classes of numbers. Such contexts are in constant use: integers are defined as classes of pairs of natural numbers;[10] rationals as classes of pairs of integers; real numbers as classes of rationals, etc. But, if a number is to be the member of a class, then it has to be an object and there is no way to explain it away. There is a possibility of explaining *classes* away, but mathematics, i.e. the standard classical mathematics, is founded on classes and does not explain them away. (After all, we are interested in what numbers are, and not interested in what numbers could be in a could-be-reconstructed-mathematics.)

This was Frege's point, too. He insisted that:

Every individual number is a self-subsistent object[11]. (Frege, 1978, p. 67e)

The definitions (d0), (d1), (d2), etc. are not definitions of the self-subsistent objects 0, 1, 2, etc. They only define the meanings of the phrases "the number 0 belongs to", "the number 1 belongs to", "the number 2 belongs to", etc. Hence, they ought to be rejected.

> It is tempting to define 0 by saying that the number 0 belongs to a concept if no object falls under it (...) the number 0 belongs to a concept, if the proposition that *a* does not fall under that concept is true universally, whatever *a* may be.
>
> Similarly, we could say: the number 1 belongs to a concept *F*, if the proposition that *a* does not fall under *F* is not true universally, whatever *a* may be, and if from the propositions
>
> "*a* falls under *F*" and "*b* falls under *F*"
>
> It follows universally that *a* and *b* are the same. (...)
>
> These definitions suggest themselves so spontaneously in the light of our previous results, that we shall have to go into the reasons why they cannot be reckoned satisfactory. (...)

> It is only an illusion that we have defined 0 and 1; in reality we have only fixed the sense of the phrases
> "the number 0 belongs to";
> "the number 1 belongs to";
> but we have no authority to pick out the 0 and 1 here as self-subsistent objects that can be recognized as the same again[12]. (Frege, 1978, pp. 67e–68e)

If numbers are not to be explained away, then what are they? We know that they belong to pure founded classes. If a and b are numbers, then they are numbers of some pure founded classes A and B:

$$a = \mathrm{num}(A), \qquad b = \mathrm{num}(B)$$

Hence, to specify what numbers are, is to define the function num on the universe of all pure founded classes. Numbers are to be the values of this function. Of course, this should be done in accordance with the criterion of number identity: num(A) = num(B) iff A and B are of the same size. "A and B are of the same size" is to be defined as "there is a one-to-one correspondence between A and B; $A \approx B$". Hence, the function num is to be defined in accordance with the following:

(NI) *Number identity* $\quad \mathrm{num}(A) = \mathrm{num}(B) \leftrightarrow A \approx B$

> Hume long ago mentioned such a means: "When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal." This opinion, that numerical equality or identity must be defined in terms of one–one correlation, seems in recent years to have gained widespread acceptance among mathematicians. But it raises at once certain logical doubts and difficulties, which ought not to be passed over without examination[13]. (Frege, 1978, pp. 73e–74e)

The difficulty is that there is no unique function num, defined on the universe of pure founded classes, which satisfies the criterion of number identity (NI). In one word, there is no unique solution of (NI).

First of all, the range of num is not determined by (NI). If we agree that values of num (i.e. numbers) should be mathematical objects and if we also agree that all mathematical objects should be restricted to pure founded classes (cf. above), then it follows that the range of num should be the universe of pure founded classes. But even then, there is no unique function num which satisfies (NI).

Namely, the $\approx$ relation (the relation of equinumerosity) is an equivalence relation on the universe of pure founded sets.[14] By a standard theorem on such relations the universe is partitioned by it into disjoint classes of mutually equinumerous sets (cf. Figure 2).

Each such equivalence class $[x]$ is the class of all sets equinumerous to x, $[x] = \{y \colon y \approx x\}$, and every function n defined on these equivalence classes induces the corresponding function num:

$$\mathrm{num}(x) = \mathrm{n}([x]),$$

which satisfies (NI). Of course, there are a lot of functions defined on the equivalence classes and therefore there are a lot of num functions.

The uniqueness of the num function could be forced by imposing further conditions on the function. Needless to say the conditions should be appropriate, just as (NI) is. The Fregean condition:

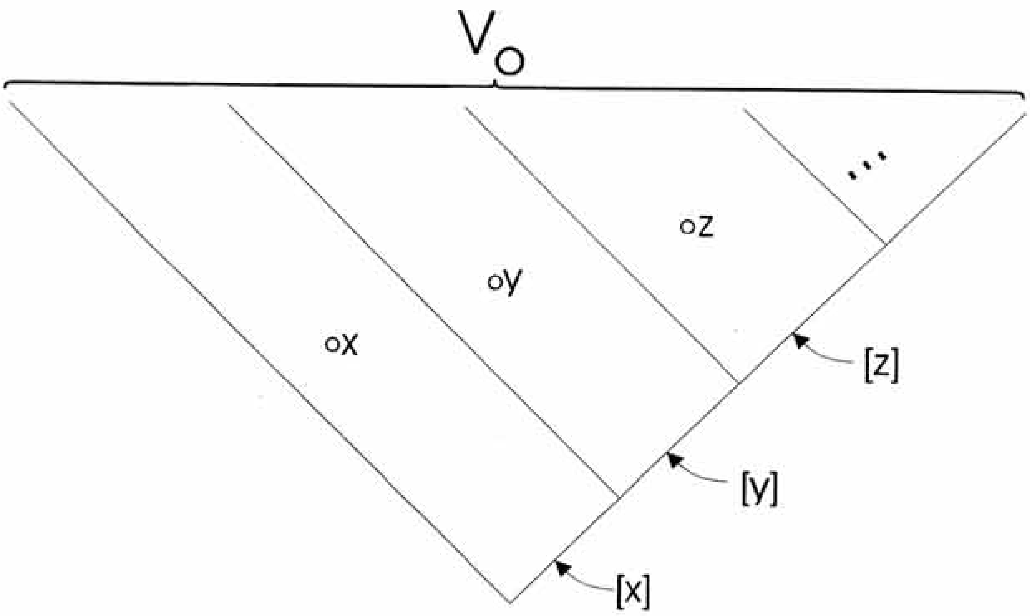

Figure 2. *The classes of mutually equinumerous sets.*

(FC) $x \in \text{num}(x)$

which postulates that each set is a member of the number ascribed to it, is one such condition.[15] There is exactly one function num which satisfies (NI) and (FC). This is the Fregean number function (cf. Figure 3):

(FN) $\text{num}(x) = [x]$

The corresponding n-function is the identity function, $\text{n}([x]) = [x]$.

The difficulty of Fregean definition (FN) is that, according to the definition, numbers are proper classes. It means that numbers could not be members of further

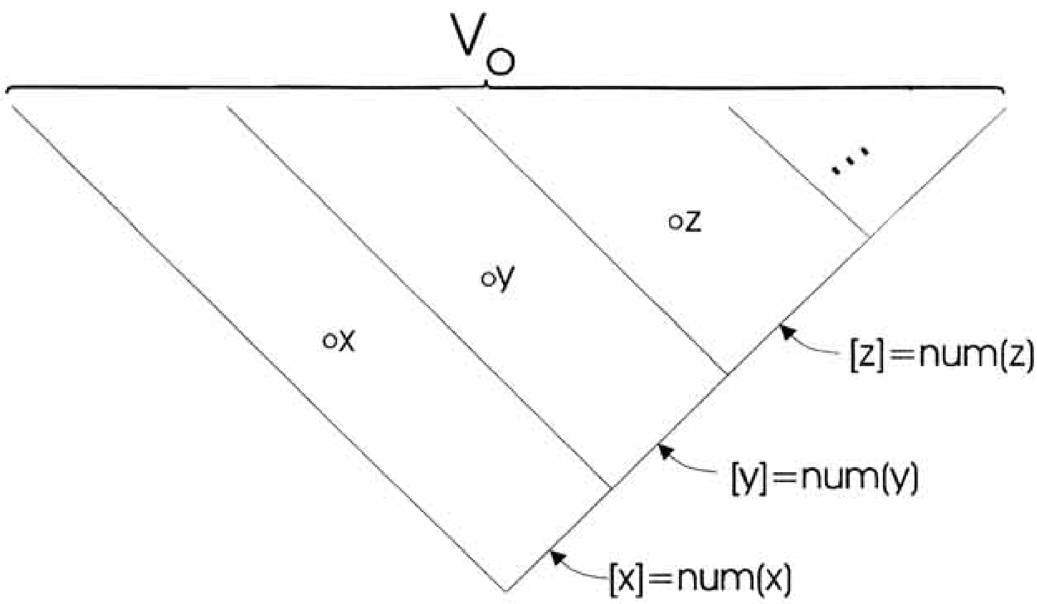

Figure 3. *Frege's numbers.*

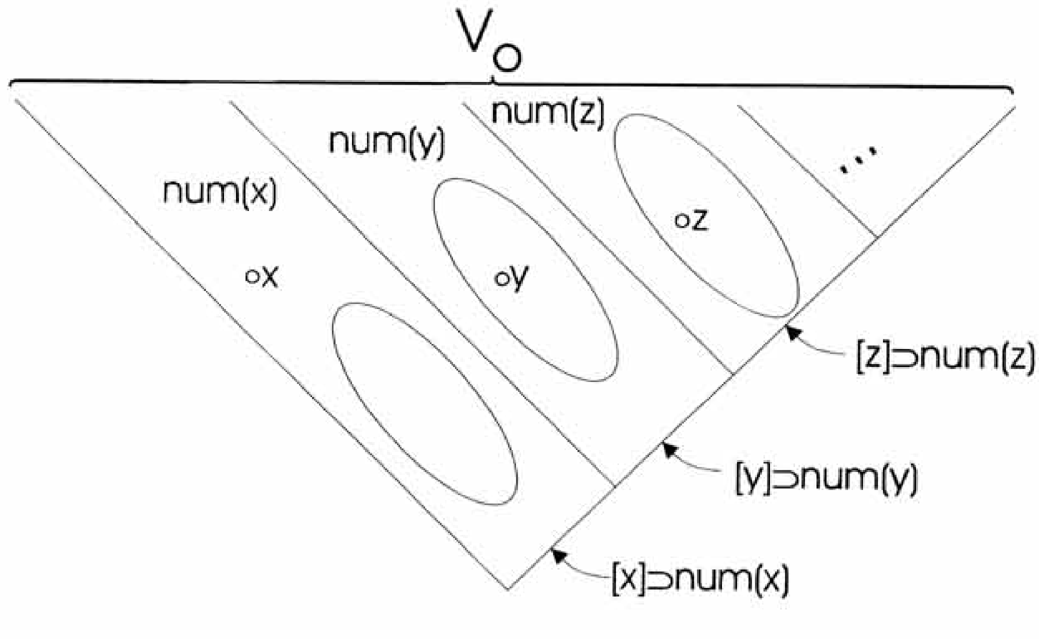

Figure 4. *Scott's numbers.*

classes. It renders the definition mathematically useless and, therefore, inadmissible (cf. above).

Scott's definition of numbers (cf. Scott, 1955):

(SN) $\text{num}(x) = [x]_{\min}$

where $[x]_{\min}$ is the subset of $[x]$ consisting of elements of least rank, also satisfies (NI). It dispenses with Frege's difficulty, because $[x]_{\min}$ is a *set* contained in $[x]$ (cf. Figure 4).

Nevertheless, it has to be abandoned, because the further condition it satisfies (that $\text{num}(x)$ should consist of elements of $[x]$ of least rank) is an *ad hoc* condition. The only motivation for using Scott's condition is to eliminate Frege's difficulty.

Another, further natural condition is that mutually equinumerous members of $[x]$ are represented by one of them. It means that num associates to any given x its *representative* from $[x]$ (cf. Figure 5):

(RC) $\text{num}(x) \in [x]$

The problem is that there are many num functions which satisfy (NI) and the representative condition (RC). A further condition is needed to separate the unique num function from all of them. Without such a condition $\text{num}(x)$ is as conventional as the Paris metre is. (The Paris metre represents all lengths of its span, just as $\text{num}(x)$ represents all sets of its numerosity.) Hence, we could not talk about numbers, but rather about *conventional* numbers.

Nevertheless, to whatever use numbers may be put, conventional numbers may be put equally well. For example, using von Neumann's conventional numbers $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$, etc. we could interpret:

(0) "The number 0 belongs to the set x."

(1) "The number 1 belongs to the set y."

(2) "The number 2 belongs to the set z, etc.

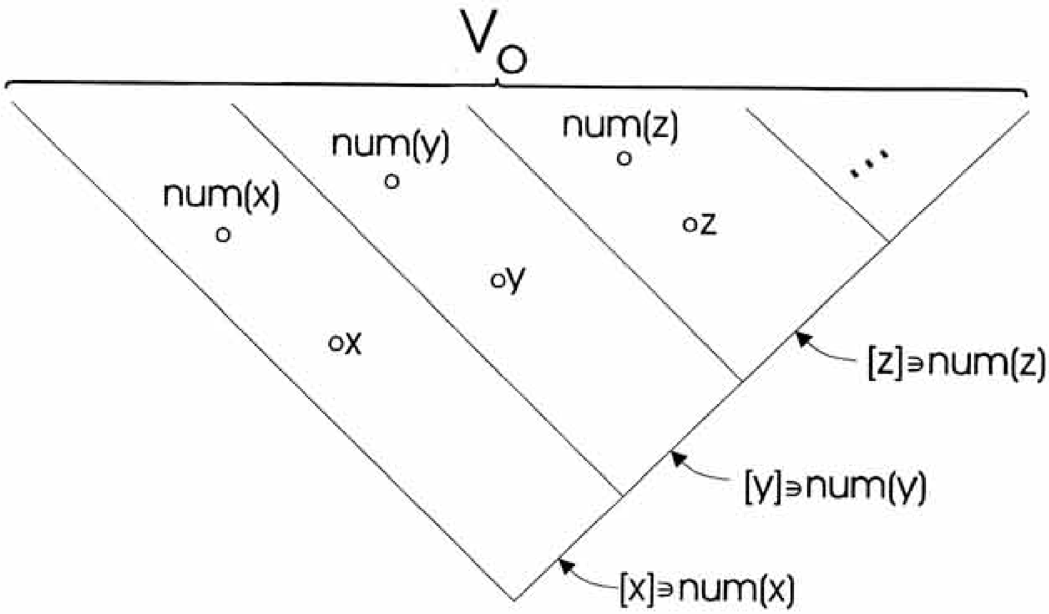

Figure 5. *The representative numbers.*

in the following way:

(c0) "$x \approx \varnothing$."

(c1) "$y \approx \{\varnothing\}$." (c2) "$z \approx \{\varnothing, \{\varnothing\}\}$, etc.

Any other conventional numbers would do just as well.

Similarly, conventional integers are easily defined as classes of pairs of conventional natural numbers;[16] conventional rationals as classes of pairs of conventional integers; conventional real numbers as classes of conventional rationals, etc.

In a sense, conventional numbers explain numbers away. But conventional numbers are objects, which are elements of further classes, and the main problem of explaining numbers away is not present now. To whatever use numbers may be put, conventional numbers may be put equally well. Mathematics has no need of numbers. Conventional numbers suffice.

If this is disappointing we could add that von Neumann's conventional numbers are not as conventional as it seems. First, von Neumann's conventional numbers are conventional ordinal numbers. As far as natural numbers are concerned there is no difference. Natural numbers are cardinal numbers of finite sets and there is always the unique well-ordering of a finite set. It means, if finite sets are of the same size then they are of the same length. On the other hand, there are many well-orderings of an infinite set. It means that infinite sets of the same size may have different lengths. Hence, to the cardinal number of an infinite set there are many corresponding ordinal numbers. The first of them is unique and it is quite appropriate to identify the cardinal number with this unique ordinal number. A cardinal number is the first ordinal number of its cardinality. Cardinal numbers, defined in this way, are as conventional as their corresponding ordinal numbers are. We would like to show that von Neumann's ordinal numbers are not as conventional as it seems.

We should start with the question: What is an ordinal number the ordinal number of? We showed that a cardinal number is the cardinal number of a pure founded class.

In the same way, we could show that an ordinal number is the ordinal number of a pure founded well-ordered class. If α and β are ordinal numbers, then they are ordinal numbers of some pure founded well-ordered classes $(A, <_A)$ and $(B, <_B)$:

$$\alpha = \mathrm{Ord}(A, <_A), \qquad \beta = \mathrm{Ord}(B, <_B)$$

Hence, to specify what ordinal numbers are is to define the function Ord on the universe of all pure founded well-ordered classes. Ordinal numbers are to be the values of this function. This should be done in accordance with the criterion of ordinal number identity: $\mathrm{Ord}(A, <_A) = \mathrm{Ord}(B, <_B)$ iff $(A, <_A)$ and $(B, <_B)$ are of the same length. "$(\mathrm{A}, <_A)$ and $(\mathrm{B}, <_B)$ are of the same length" is to be defined as "there is a one-to-one correspondence between A and B which is order preserving; $(A, <_A) \sim (B, <_B)$". Hence, the function Ord is to be defined in accordance with the following:

(ONI) *Ordinal number identity* $\mathrm{Ord}(A, <_A) = \mathrm{Ord}(B, <_B) \leftrightarrow (A, <_A) \sim (B, <_B)$

The difficulty is that there is no unique function Ord which satisfies (ONI). Even if we agree that values of Ord should be pure founded well-ordered classes, there is no unique Ord which satisfies (ONI).

The uniqueness of the Ord function could be forced by imposing further conditions on the function. A further natural one is that mutually equi-ordered sets[17] are represented by one of them:

(ORC) $\mathrm{Ord}(A, <_A) \in [(A, <_A)]$

The problem is that there are still many Ord functions which satisfy (ONI) and (ORC). Further conditions are needed to separate the unique Ord function from all of them. Without such conditions, $\mathrm{Ord}(A, <_A)$ is completely conventional.

There are such conditions which are quite appropriate. An ordinal number $\mathrm{Ord}(A)$ is the ordinal number of a well-ordered class A, i.e. $\mathrm{Ord}(A)$ is the length of A.[18] But, the primary meaning of ordinal numbers is that they are the ordinal numbers of the members of a well-ordered class. The well-ordered class A has its 5th member, its 157th member and, perhaps, its ωth member and its ε_0th member. The ordinal number of a in A is denoted by $\mathrm{ord}(a)$[19] and it is the distance of a in A. The connection between 'small' ord and 'big' Ord is straightforward. The distance of a in A is the length of $\{x: x < a\}$, i.e.

(oO) $\mathrm{ord}(a) = \mathrm{Ord}\{x: x < a\}$[20]

On the other hand the length of $\{x: x < a\}$ is completely determined by the set of distances $\{\mathrm{ord}\ x: x < a\}$, i.e.

(Oo) $\mathrm{Ord}\{x: x < a\} = \{\mathrm{ord}(x): x < a\}$

is an appropriate condition.
From (oO) and (Oo) it follows:

$$\mathrm{ord}(a) = \mathrm{Ord}\{x: x < a\} = \{\mathrm{ord}(x): x < a\}$$

From the principle of transfinite recursion on well-ordered classes (cf. above), it follows that for any well-ordered class $(A, <)$ there is the unique function ord which satisfies $\mathrm{ord}(a) = \{\mathrm{ord}(x): x < a\}$. Therefore, there is the unique function Ord which satisfies (ONI), (ORC), (oO) and (Oo).[21] This is illustrated with the following example.

$$(A, <) = (a < b < c)$$

$$\text{ord}(a) = \{\text{ord}(x)\colon x < a\} = \varnothing$$

$$\text{ord}(b) = \{\text{ord}(x)\colon x < a\} = \{\text{ord}(a)] = \{\varnothing\}$$

$$\text{ord}(c) = \{\text{ord}(x)\colon x < c\} = \{\text{ord}(a), \text{ord}(b)] = \{\varnothing, \{\varnothing\}\}$$

$$\text{ord}(A) = \{\text{ord}(x)\colon x \in A\} = \{\varnothing, \{\varnothing], \{\varnothing, \{\varnothing\}\}\}$$

We may conclude: if (oO) and (Oo) are appropriate conditions (and we believe they are) then Ord is not conventional and, therefore, von Neumann's numbers are not just conventional numbers. They are the numbers.

Notes

1. As blieb noch die Frage, von wem durch eine Zahlangabe etwas ausgesagt werde. (Frege, 1884, p. 58.)
2. Wenn ich in Ansehung derselben äussern Erscheinung mit derselben Wahrheit sagen kann: "dies ist ein Baumgruppe" und "dies sind fünf Bäume" oder "hier sind vier Compagnien" und "hier sind 500 Mann," so ändert sich dabei weder das Einzelne noch das Ganze, das Aggregat, sondern meine Benennung. Das ist aber nur das Zeichen der Ersetzung eines Begriffes durch einen andern. Damit wird uns als Antwort auf die erste Frage des vorigen Paragraphen nahe gelegt, dass die zahlangabe eine Aussage von einem Begriffe enthalte. Am deutlichsten ist dies vielleicht bei der Zahl 0. Wenn ich sage: "die Venus hat 0 Monde", so ist gar kein mond oder Aggregat von Monden da, von dem etwas ausgesagt werden könnte; aber dem Begriffe "Venusmond" wird dadurch eine Eigenschaft beigelegt, nämlich die, nichts unter sich zu befassen. Wenn ich sage: "der Wagen des Kaisers wird von vier Pferden gezogen," so lege ich die Zahl vier dem Begriffe "Pferd, das den Wagen des Kaisers zieht," bei. (Frege, 1884, p. 59.)
3. Dass eine Zahlangabe etwas Thatsächliches von unserer Auffassung Unabhä ngiges ausdrückt, kann nur den Wunder nehmen, welcher den Begriff für etwas Subjectives gleich der Vorstellung hält. Aber diese Ansicht ist falsch. Wenn wir z. B. den Begriff des Körpers dem des Schweren oder den des Wallfisches dem des Säugethiers unterordnen, so behaupten wir damit etwas Objectives. Wenn nun die Begriffe subjectiv wäre auch die Unterordnung des einen unter den andern als Beziehung zwischen ihnen etwas Subjectives wie eine Beziehung zwischen Vorstellungen. (Frege, 1884, p. 60.)
4. Ich unterscheide das Objective von dem Handgreiflichen, Räumlichen, Wirklichen. Die Erdaxe, der Massenmittelpunkt des Sonnensystems sind objectiv, aber ich möchte sie nicht wirklich nennen, wie die Erde Selbst. (Frege, 1884, p. 35)
5. It is meant here, that the English word "object" has the same meaning as the German word "gegenstand". Incidentally, it is sometimes confusing that Austin, in his translation (Frege, 1978), used one English word "object" as translation of two german words "gegenstand" and "object".
6. Even Frege (1884, p. 80) wrote: Ich glaube, dass für "Umfang des Begriffes" einfach "Begriff" gesagt werden könte. (I believe, it could be simply said "concept" instead of "extension of concept".)
7. The answer to our question (what classes are there) looks very simple. Nevertheless, mathematicians struggled for it for more then 70 years. Those who were logically inclined wanted to base everything on the presupposedly logical principles of comprehension and extensionality and that was not possible. Those who were more mathematically minded did not care for that. But they were not successful explaining (with no use of the independent notion of ordinal number) that the universe of all classes V[Ind] is just the cumulative hierarchy $U\{V_\alpha[\text{Ind}]\colon \alpha \in O\}$. Von Neumann was the first one to succeed.
8. For the meaning of "the same size" cf. below.
9. Perhaps, we could say that mathematics is founded on nothing.
10. Natural numbers are cardinal numbers of finite classes. Ordinal numbers are discussed later.
11. Jede einzelne Zahl ist ein selbständiger Gegenstand. (Frege, 1884, p. 67.)
12. Es liegt nahe zu erklaren: einem Begriffe kommt die Zahl 0 zu, wenn kein Gegenstand unter ihn fällt (... einem Begriffe kommt die Zahl 0 zu, wenn allgemein, was auch *a* sei, der Satz gilt, dass *a* nicht unter diesen Begriff falle.

    In ähnlicher Weise könnte man sagen: einem Begriffe F kommt die Zahl 1 zu, wenn nicht allgemein, was auch *a* sei, der Satz gilt, dass *a* nicht unter F falle, und wenn aus des Sätzen

    "*a* fällt unter *F*" und "*b* fällt unter *F*"

    allgemein folgt, dass *a* und *b* dasselbe sind. (...)

Diese Erklarungen bieten sich nach unsern bisherigen Ergebnissen so ungezwungen dar, dass es einer Darlegung bedarf, warum sie uns nicht gen ügen können. (...)
Es ist nur Schein, dass wir die 0, die 1 erklart haben; in Wahrheit haben wir nur den Sinn der Redensarten

"die Zahl Ø kommt zu",
"die Zahl 1 kommt zu"

festgestellt; aber es nicht erlaubt, hierin die 0, die 1 als selbständige, wiedererkennbare Gegenstände zu unterscheiden. (Frege, 1884, pp. 67–68.)

13. Ein solches Mittel nennt schon Hume: "Wenn zwei Zahlen so combinirt werden, dass die eine immer eine Einheit hat, die jeder Einheit der andern entspricht, so geben wir sie als gleich an." Es scheint in neuerer Zeit die Meinung unter den Mathematikern vielfach Anklang gefunden zu haben, dass die Gleichheit der Zahlen mittels der eindeutigen Zuordnung definirt werden musse. Aber es erheben sich zunächst logische Bedenken und Schwierigkeiten, an denen wir nicht ohne Prüfung vorbeigehen dürfen. (Frege, 1884, pp. 73–74.)
14. We may restrict ourselves to sets because the proper classes are mutually equinumerous to one another. Hence, one and the same number belongs to all proper classes and it could be dealt with separately.
15. Incidentally, (FC) precludes the proper classes to have a number, because a proper class could not be a member of any class.
16. Conventional natural numbers are conventional cardinal numbers of finite classes. Conventional ordinal numbers are discussed later.
17. We may restrict ourselves to well-ordered sets, because all proper well ordered classes are of the same ordinal type. Hence, one and the same ordinal number belong to all of them and it could be dealt with separately.
18. Well-ordered class A" is short for "well-ordered class $(A, <_A)$".
19. The function ord is always associated with some well-ordered class A, on which it is defined. Strictly, we should write ord_A. If it is clear which A it is, we usually omit A.
20. (oO) is the definition of ord in terms of Ord. The criterion of ordinal number identity for ord:

    (oNI) $\quad \mathrm{ord}_A(a) = \mathrm{ord}_B(b) \leftrightarrow \{x: x <_A a\} \sim \{y: y <_B b\}$

    follows from (oO) and (ONI). Namely, $\mathrm{ord}(a) = \mathrm{ord}(b)$ iff $\mathrm{Ord}\{x: x < a\} = \mathrm{Ord}\,\{y: y < b\}$ iff $\{x: x < a\} \sim \{y: y < b]$.
21. Ord is defined on well-ordered sets. The ordinal number of any proper well-ordered class is O, i.e. the proper class of all ordinal numbers of well-ordered sets (cf. 17)).

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