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A NOTE ON SUM-ELIMINATOR

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Abstract. We fill in a gap in Takeuti's constructive demonstration of the fact that: Whenever a concrete method of constructing a decreasing sequences of ordinals less than ε_0 is given, any such decreasing sequence must be finite. We have proved a result on sum-eliminator, which fills the gap.

In Ch. 2.11 of Takeuti's »Proof theory« it is demonstrated that:

Whenever a concrete method of constructing decreasing sequence of ordinals is given, any such decreasing sequence must be finite.

In step (XII) it is demonstrated that we can concretely construct $(\mu \cdot m, n+1)$-eliminator from given $(\mu, n+1)$-eliminator, for every $m < \omega$. This particular demonstration works unfortunately for $n > 0$ only, what makes it incomplete. To fill the gap we have to demonstrate:

Given an α_1-eliminator (it means $(\alpha_1, 1)$-eliminator) and an α_2-eliminator we can construct an $\alpha_1 + \alpha_2$-eliminator.

Suppose α_1-eliminator A_1 and α_2-eliminator A_2 are given. In order to construct $\alpha_1 + \alpha_2$-eliminator we give a concrete method B.

Given a descending sequence $a_0 > a_1 > \ldots$, the method B should produce a decreasing $\alpha_1 + \alpha_2$-sequence $b_0 > b_1 > \ldots$, which satisfies the condition: b_0 is $\alpha_1 + \alpha_2$-major part of a_0 and if $b_0 > b_1 > \ldots$ is a finite sequence, then so is $a_0 > a_1 > \ldots$.

The prescription for B is as follows. Suppose we are given a decreasing sequence $a_0 > a_1 > \ldots$. Then α_1-eliminator A_1 can be applied to this sequence and hence it concretely produces an α_1-sequence $c_0 > c_1 > \ldots$. Each c_i can be written as

$$c_i = c_i' + d_i, \qquad c_i' \text{ is } \alpha_1 + \alpha_2\text{-major part of } c_i. \tag{1}$$

Put $b_0 = c_0'$. Suppose $b_0 > b_1 > \ldots > b_n$ have been constructed and $b_n = c_{p_n}'$. We proceed as follows. If $c_{p_n}' = c_{p_n+1}' = \ldots = c_{p_n+k}'$ and c_{p_n+k}' is the last term in the sequence, then stop. Otherwise $c_{p_n}' = c_{p_n+1}' = \ldots = c_{p_n+l_{n+1}-1}' > c_{p_n+l_{n+1}}'$ for some l_{n+1}. This is true since

$$c_{p_n}' = c_{p_n+1}' = \ldots \tag{2}$$

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together with (1) implies

$$d_{p_n} > d_{p_n+1} > \dots . \quad (3)$$

Each d_i $(i > p_n)$ can be written in the canonical form

$$d_i = \omega^{\gamma_1^i} + \dots + \omega^{\gamma_{n_i}^i}. \quad (4)$$

But $c_0 > c_1 > \dots$ is an α_1-sequence and this together with (1) implies

$$\alpha_1 \leqslant \gamma_l^i < \alpha_1 + \alpha_2 \qquad \text{for } l = 1, \dots, n_i. \quad (5)$$

It follows that we can find ordinals β_l^i such that

$$\gamma_l^i = \alpha_1 + \beta_l^i \quad \text{and} \quad \beta_l^i < \alpha_2. \quad (6)$$

From (4) and (6) it follows that

$$d_i = \omega^{\alpha_1 + \beta_1^i} + \dots + \omega^{\alpha_1 + \beta_{n_i}^i} = \omega^{\alpha_1} (\omega^{\beta_1^i} + \dots + \omega^{\beta_{n_i}^i}) = \omega^{\alpha_1} \cdot d_i'. \quad (7)$$

$$d_i' < \omega^{\alpha_2}.$$

From (3) we have $d'_{p_n} > d'_{p_n+1} > \dots$. Applying α_2-eliminator A_2 to this sequence (for which $d'_{p_n} < \omega^{\alpha_2}$), as described in step (VIII) of Takeuti's demonstration, we conclude that the given sequence is finite. Therefore the sequence (2) is also finite. Put now $b_{n+1} = c'_{p_n+l_{n+1}}$. From the definition, it is obvious that $b_0 > b_1 > \dots > b_k > \dots$ is decreasing sequence. Suppose this sequence is finite, say $b_0 > b_1 > \dots > > b_k$. Then according to the prescribed construction of b_0 and b_{n+1} the original sequence $a_0 > a_1 > \dots$ is finite and has exactly $p_k = l_0 + + l_1 + \dots + l_k$ members. Thus the method B is an $\alpha_1 + \alpha_2$-eliminator.

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NAPOMENA O ELIMINATORU SUME

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Sadržaj

U Takeutijevom konstruktivnom dokazu tvrdnje: »Svaki konkretnom metodom konstruiran i strogo silazan niz ordinala manjih od ε_0 mora biti konačan.«, nalazimo netrivijalni propust koji dokaz čini nepotpunim. U radu je dokazana tvrdnja o suma -eliminatoru koja popunja spomenutu prazninu.