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A PROOF OF THE CHARACTERIZATION THEOREM FOR CONSEQUENCE RELATIONS

by ZVONIMIR ŠIKIĆ in Zagreb (Yugoslavia)

We consider the multiple consequence relation as defined in [2]. The definition is there prompted by the following line of argument: To say that a set of conclusions follows from a given set of premisses is to say that at least one of the conclusions must be true if the premisses are all true. That means that each possible state of affairs in which all the premisses are true is one in which some of the conclusions are true. Assuming the formulae of our language L to be capable of truth and untruth, each relevant state of affairs is represented by the partition (T, U) of formulae of L such that the formulae of T are true in this state of affairs, while the formulae of U are untrue in it. If $\mathcal{P}$ is the set of all partitions which correspond to the possible state of affairs, it is plausible to define the consequence relation with regard to $\mathcal{P}$ as follows.

Definition 1. Let X and Y be sets of formulae of L, i.e. $X \subseteq L$ and $Y \subseteq L$. Y is a *consequence of X with regard to $\mathcal{P}$ ($X \vdash_{\mathcal{P}} Y$)* iff there is no $(T, U) \in \mathcal{P}$ such that $X \subseteq T$ and $Y \subseteq U$. It is also said that the set of partitions $\mathcal{P}$ *generates* the consequence relation $\vdash_{\mathcal{P}}$.

If we presuppose nothing about the internal structure of the formulae of L and about their semantical interconnections, we have to be prepared to allow any set of partitions of formulae of L to play the role of $\mathcal{P}$. So, we are led to the general definition of the (multiple) consequence relation proposed in [2].

Definition 2. A relation $\vdash$ on the partitive set of L is a *consequence relation* iff there is a set of partitions $\mathcal{P}$ such that $\vdash = \vdash_{\mathcal{P}}$.

It is easy to prove that $(T, U) \in \mathcal{P}$ iff $T \nvdash_{\mathcal{P}} U$. Hence, the set $\mathcal{P}$ of partitions which generates a (multiple) consequence relation $\vdash$ is completely determined by $\vdash$, i.e.

(1) $$\mathcal{P} = \{(T, U) : T \cup U = L \;\&\; T \cap U = \emptyset \;\&\; T \nvdash U\}.$$

(This is in sharp contrast with the fact that the set of partitions which generates a single-conclusion consequence relation is *not* uniquely determined with the relation.)

A special case of consequence relation, i.e. the compact case, is considered already in [1]. It is proved there that the well known "structural rules":

(Overlap) $X \nvdash Y \Rightarrow X \cap Y = \emptyset$,

(Dilution) $X \subseteq X' \;\&\; Y \subseteq Y' \;\&\; X' \nvdash Y' \Rightarrow X \nvdash Y$,

(Cut) $X \nvdash Y \Rightarrow X \nvdash Y, A$ or $A, X \nvdash Y$

(where "," has usual meaning of union) characterize compact consequence relations. Moreover, we could say that SCOTT discovered an explicit definition of the consequence relation, which is implicitly defined with the well-known structural rules, thus proving the completeness of the rules.

The characterisation theorem for (general) consequence relations is proved in [2]. It is proved there that a relation $\vdash$ is a consequence relation if and only if the following conditions are fulfilled:

(Overlap) $X \nvdash Y \Rightarrow X \cap Y = \emptyset$,

(Dilution) $X \subseteq X' \,\&\, Y \subseteq Y' \,\&\, X' \nvdash Y' \Rightarrow X \nvdash Y$,

(Cut for sets) $X \nvdash Y \Rightarrow \forall Z \exists Z_1 \exists Z_2 (Z_1 \cup Z_2 = Z \,\&\, Z_1 \cap Z_2 = \emptyset \,\&\, X, Z_1 \nvdash Y, Z_2)$.

But we could not say that SHOESMITH and SMILEY discovered explicit definition of the consequence relation, which is implicitly defined with the well-known overlap, dilution and cut for sets, because the cut for sets had not been known before. As a matter of fact, SHOESMITH and SMILEY consider several rivals to cut for sets (cf. cut_1, cut_2 and cut_3 on p. 32 of [2]), concluding finaly that cut for sets is the right choice just because "only it is strong enough to sustain (the characterisation) theorem".

Our purpose is to prove that the conjunction of the simple special cases of overlap, dilution and cut for sets, namely the conjunction of

(Overlap for L) $X \cup Y = L \,\&\, X \nvdash Y \Rightarrow X \cap Y = \emptyset$,

(Dilution for L) $\exists T \exists U (T \cup U = L \,\&\, T \cap U = \emptyset \,\&\, X \subseteq T \,\&\, Y \subseteq U \,\&\, T \nvdash U) \Rightarrow X \nvdash Y$,

(Cut for sets for L) $X \nvdash Y \Rightarrow \exists T \exists U (T \cup U = L \,\&\, T \cap U = \emptyset \,\&\, X, T \nvdash Y, U)$

is nothing else but a trivial preformulation of Definition 2. This makes the caracterization theorem quite a natural result, and cut for sets (for L) quite a natural choice.

We know, starting from Definition 2, that $\vdash$ is a consequence relation iff there is $\mathcal{P}$ such that $\vdash = \vdash_{\mathcal{P}}$. Hence, it follows from (1) that

(2) $\vdash$ is a consequence relation iff $\vdash = \vdash_{\mathcal{P}}$ for $\mathcal{P}$ in (1).

There follows a sequence of trivial preformulations of the right side of (2):

$\forall X \forall Y (X \vdash Y \Leftrightarrow X \vdash_{\mathcal{P}} Y)$

iff $\forall X \forall Y (X \nvdash Y \Leftrightarrow X \nvdash_{\mathcal{P}} Y)$

iff $\forall X \forall Y (X \nvdash Y \Leftrightarrow \exists (T, U) \in \mathcal{P} (X \subseteq T \,\&\, Y \subseteq U))$

iff $\forall X \forall Y (X \nvdash Y \Leftrightarrow \exists T \exists U (T \cup U = L \,\&\, T \cap U = \emptyset \,\&\, X \subseteq T \,\&\, Y \subseteq U \,\&\, T \nvdash U)$

iff (Overlap-Cut_L) & (Dilution_L),

where

(Overlap-Cut_L) $\forall X \forall Y (X \nvdash Y \Rightarrow \exists T \exists U (T \cup U = L \,\&\, T \cap U = \emptyset \,\&\, X \subseteq T \,\&\, Y \subseteq U \,\&\, T, X \nvdash U, Y))$,

(Dilution_L) $\forall X \forall Y (\exists T \exists U (T \cup U = L \,\&\, T \cap U = \emptyset \,\&\, X \subseteq T \,\&\, Y \subseteq U \,\&\, T \nvdash U) \Rightarrow X \nvdash Y)$.

When we consider the sets X and Y such that $X \cup Y = L$, we get Overlap for L as a trivial consequence of Overlap-Cut_L, and also, we immediately realize that, in the presence of Overlap for L, Overlap-Cut$_L$ is equivalent to Cut for L (namely, $X \subseteqq T$ and $Y \subseteqq U$ follow from $(T \cup X) \cap (U \cup Y) = \emptyset$).

Hence, by trivialy preformulating Definition 2, we proved the characterization theorem for consequence relations in the following form.

Theorem. *A relation* $\vdash$ *is a consequence relation iff* $\vdash$ *fulfils* (Overlap for L), (Dilution for L) *and* (Cut for sets for L).

The characterization theorem of Shoesmith and Smiley is a simple corollary of our theorem, because (Overlap for L), (Dilution for L) and (Cut for sets for L) are special cases of (Overlap), (Dilution) and (Cut for sets).

References

[1] Scott, D., Completeness and axiomatizability in many-valued logic. In: Proceedings Symposia Pure Mathematics, American Mathematical Society, **25** (1974), pp. 411–435.
[2] Shoesmith, D. J., and T. J. Smiley, Multiple-Conclusion Logic. Cambridge, 1978.

Z. Šikić
FSB
D. Salaja 5
41000 Zagreb
Yugoslavia

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