MATHEMATICS, PHYSICS AND MUSIC – A CASE STUDY

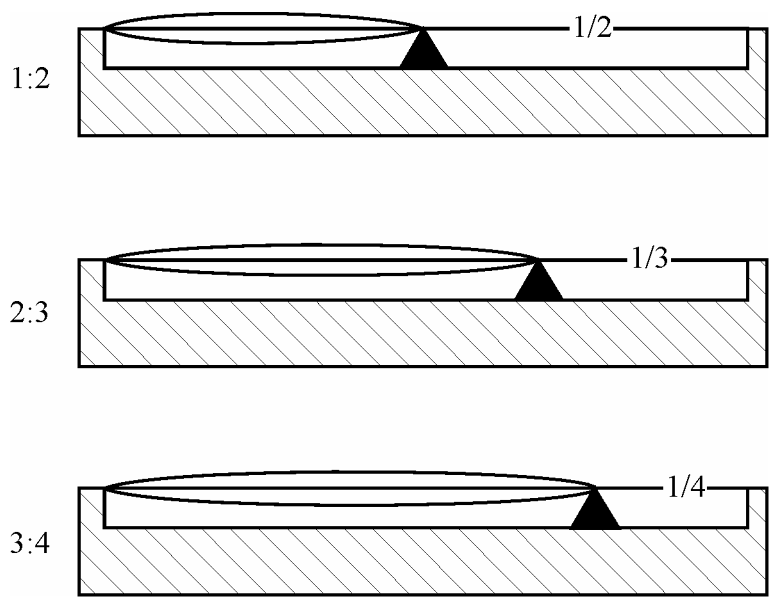
Z. Šikic, Zagreb

Abstract:

I discuss the Pythagorean law of small numbers and its use in interpretations of our sensory discriminations of consonance vs. dissonance. It seems that the fact of non-western musical traditions contradicts the law and forces us to interpret the discriminations as acquired and subjective. I would like to show that this is a wrong interpretation, because it is based on the irrelevant empirical evidence. It does not take into account the correct mathematical and physical explanation of the law, provided by Helmholtz's theory in 1877 and corroborated by Plomp-Levelt experiment in 1965.

1. THE PROBLEM

The Pythagoreans came to believe that principles of mathematics are the principles of everything. The starting point of this rather general belief was their discovery of *"the law of small numbers"* i.e. their discovery that the pitch of a string is simply related to its length. When the length is shortened in ratio 1:2 the pitch jumps up an octave, when shortened in ratio 2:3 it jumps up a fifth, in ratio 3:4 it jumps up a fourth, in ratio 4:5 a major third etc.



To shorten the length is to enlarge the frequency and we could say that the Pythagoreans discovered that the frequency ratio between the octave and the fundamental is 2:1, between the fifth and the fundamental 3:2, between the fourth and the fundamental 4:3 etc.

Pythagoreans proceeded to describe the whole universe in terms of simple harmonic relationships; from the harmonious or inharmonious resonances in human bodies bellow the moon, to the harmony of the spheres above. To use the nomenclature of a later era, *musica instrumentals*, the ordinary music made by plucking the lyre, was extended from *musica humana* to *musica mundana*.

What interests us here is the following question. Do our discriminations of consonant and dissonant intervals have some basic origin in facts “out there” in the real world?

According to the law of small numbers it seems that there is something *unique* “out there” which we discriminate as consonance “in here”. This *unique* source of our discriminations is the harmonic sequence of frequencies $1f : 2f : 3f : 4f : \ldots$. This sequence of integer frequencies is different from all the other non-integer sequences and we perceive this *objective* difference as consonance. In this sense our discriminations are *objective* and not subjective.

This objective explanation of the consonance discriminations is the common opinion in western arts and sciences. We illustrate it with a quotation from 17th century scientist:

> *The laws of music are unchangeably fixed by nature, hence they should hold not only for the entire earth, but for the inhabitants of other planets as well.* (C. Huygens as quoted in [P]).

and a 20th century artist:

> *A music – whether folk, pop, …, tonal, atonal, …, past, future, … - all of it has a common origin in the universal phenomenon of the harmonic series.* (L. Bernstein in [B]).

But there is a huge problem with the common opinion. It is the existence of non-western musical traditions whose consonant intervals have nothing to do with the harmonic series. For example, the gamelan percussive orchestra, which is the indigenous musical tradition of Java and Bali, use 5 tone *slendro* and 7 tone *pelog* scales. Neither scale lies even remotely close to the western harmonic scales. Their consonances are based on non-integer sequences of frequencies.

Hence, there is *nothing unique* “out there” which humans discriminate as consonances “in here”. It seems that our discriminations are *subjective* and not objective.

We have two opposing conclusions. According to the law of small numbers Pythagorean just intonation, which is based on the integer sequence of frequencies, is a human universal. If we take into account the existence of the non-western musical traditions, whose scales are based on many different non-integer sequences of frequencies, it is not a human universal[1)].

2. ANOTHER DIMENSION

There is also another dimension of the problem. Do our discriminations depend on innate systems, as we have tacitly presupposed until now, or do they depend on our experiences? In other words, are these discriminations *innate* or *acquired* [2)] ? According to the law of small numbers it seems that they are innate and objective. According to the fact of the existence of different musical traditions they could be acquired and subjective or perhaps innate and subjective[3)]. Prevailing opinion is that they are acquired and that means changeable. Around this opinion evolved a lot of music-policy nonsense:

Musical racist imperialism:
The music of other cultures should evolve towards western higher forms which are based on immutable laws of nature.

Musical cultural imperialism:
The music of other cultures should evolve towards OUR higher forms which are produced by OUR superior culture.

Musical cosmopolitanism:
All musical traditions are equally worthy and should influence each other.

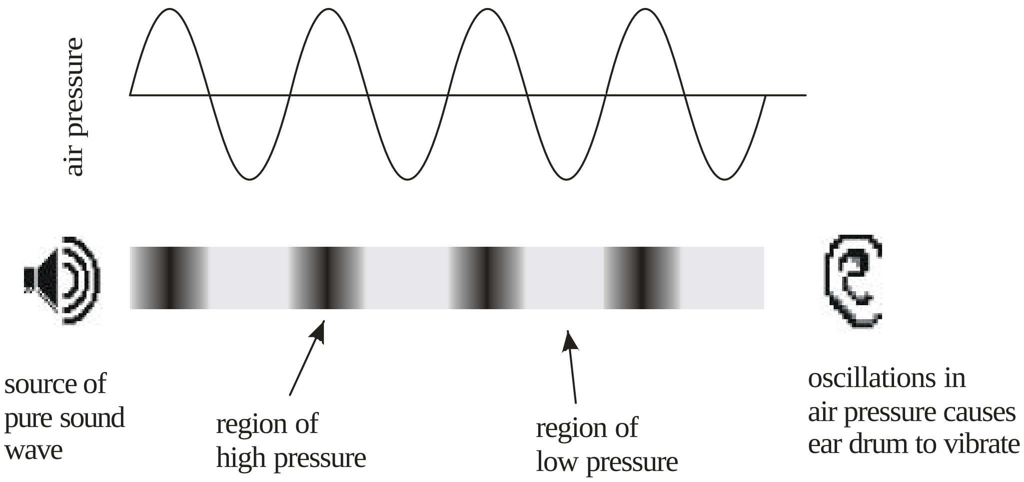
Musical nationalism:
It is OUR music and we do not want any influences.

As far as it is based on the notion of the consonant intervals it is all wrong, because it is based on the irrelevant empirical evidence. In particular it does not take into account what happened to the law of small numbers in the last few thousand years and to the understanding of the other musical traditions in the last century. Let me explain.[4)]

3. GALILEO'S THEORY

Notice that Pythagoreans offered no explanation of the law of small numbers. To offer one you should have some ideas about sound.

If you focus on perceptual aspect, sound is the sensation stimulated in the organs of hearing by vibrations in the air with frequencies in the range of 20 to 20 000 Hz. The vibrations are vibrations of a pressure wave, also known as a sound wave. It is explained in the following figure.



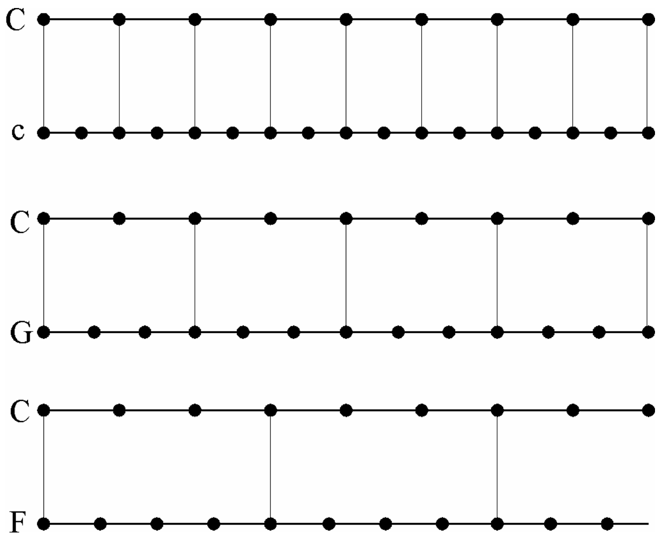
The peaks represent times when air molecules are clustered, causing higher pressure. The valleys represent times when the air density, and hence the pressure, is lower. The wave pushes against the ear drum in times of higher pressure, and pulls during times of low pressure, causing the drum to vibrate. These vibrations are perceived as sound.

In accordance with this general idea Galileo offered one of the first explanations of the law of small numbers.

> *… agreeable consonances are pairs of tones which strike the ear with a certain regularity; this regularity consists in the fact that the pulses delivered by the two tones, in the same interval of time, shall be commensurable in number, so as not to*

keep the eardrum in perpetual torment, bending in two different directions in order to yield to the ever discordant impulses [G] .

Galileo’s pulses are the periods of the corresponding sound waves. If we represent them as below, then the number of points per unit interval represents the corresponding frequency.



Looking at this representation we could say that one half of the octave is contained in the fundamental, and that explains the intimacy of the octave and its fundamental. Similarly, one third of the fifth is contained in the fundamental, and that explains a bit less intimacy of the fifth and its fundamental. In the same way one fourth of the fourth is contained in the fundamental, one fifth of the major third, one sixth of the minor third etc. That explains their diminishing consonances.

The some pattern could be represented arithmetically.

C		2		4		6		8		10		12	…
c	1	2	3	4	5	6	7	8	9	10	11	12	…

C		3		6		9		12		15		18	…
G	2		4	6	8		10	12	14		16	18	…

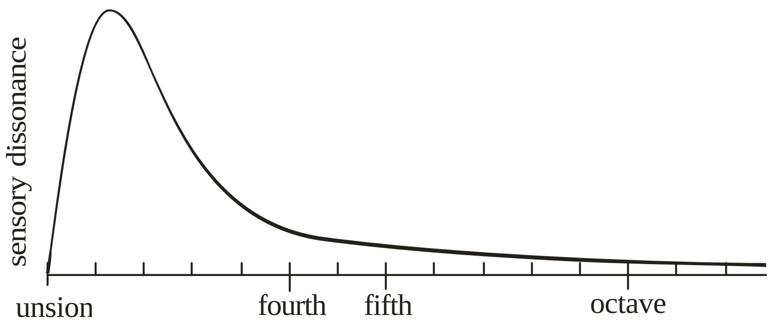
C		4		8		12		16		20		24	…
F	3		6		9	12	15		18		21	24	…

Note how initial ratios determine the intimacy of the tones in given intervals i.e. their consonances.

Galileo's theory is very nice and frequently cited, even today, but there is one big problem with it. It is not true.

4. THE TRUE THEORY FOR SIMPLE SOUNDS

In an important experiment in 1965, Plomp and Levelt investigated how untrained listeners judge the dissonance[5)] of a variety of intervals *when sounded by pairs of pure sine waves*. The result of the experiment is represented by the dissonance curve.



(1) The dissonance is minimum, zero, when both sine waves are of the same frequency.
(2) It increases rapidly to its maximum somewhere around the second, in the middle range.
(3) Then it decreases steadily back toward zero.

Notice that major 7th and minor 9th are almost indistinguishable from the octave in terms of sensory dissonance for pure sine waves. This is in complete disagreement with Galileo's theory.

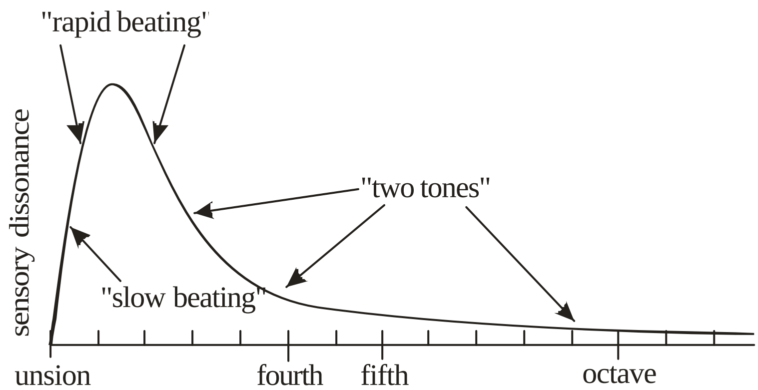
Helmholtz explained what is happening here, almost a century before Plomp and Levelet made their experiment.[6)]

(1) When the sine waves are very close in frequency they are heard as a simple pleasant tone with slow vibrations in loudness. The physical origin of this pleasant vibrato is the phenomenon of beating.[7)]

(2) Somewhat further apart in frequency the beating becomes rapid and this is heard as dissonance.

(3) Then the tones separate and are perceived individually as a consonant pair.

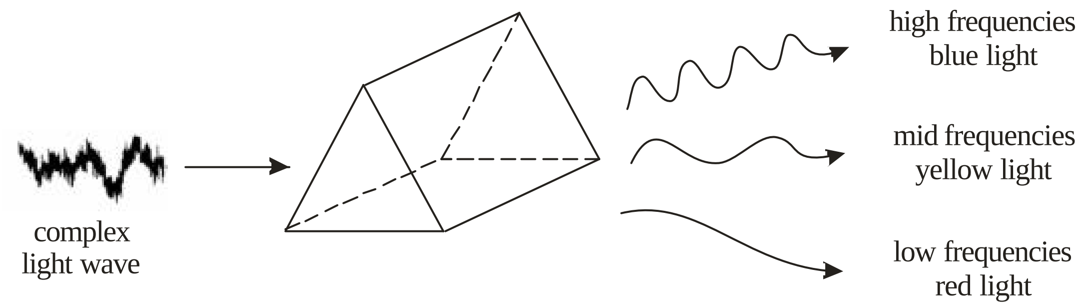
It is illustrated in the following figure.

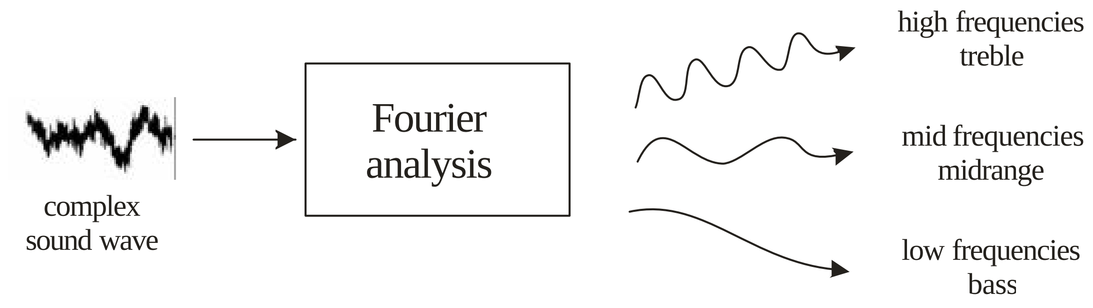


5. COMPLEX SOUNDS

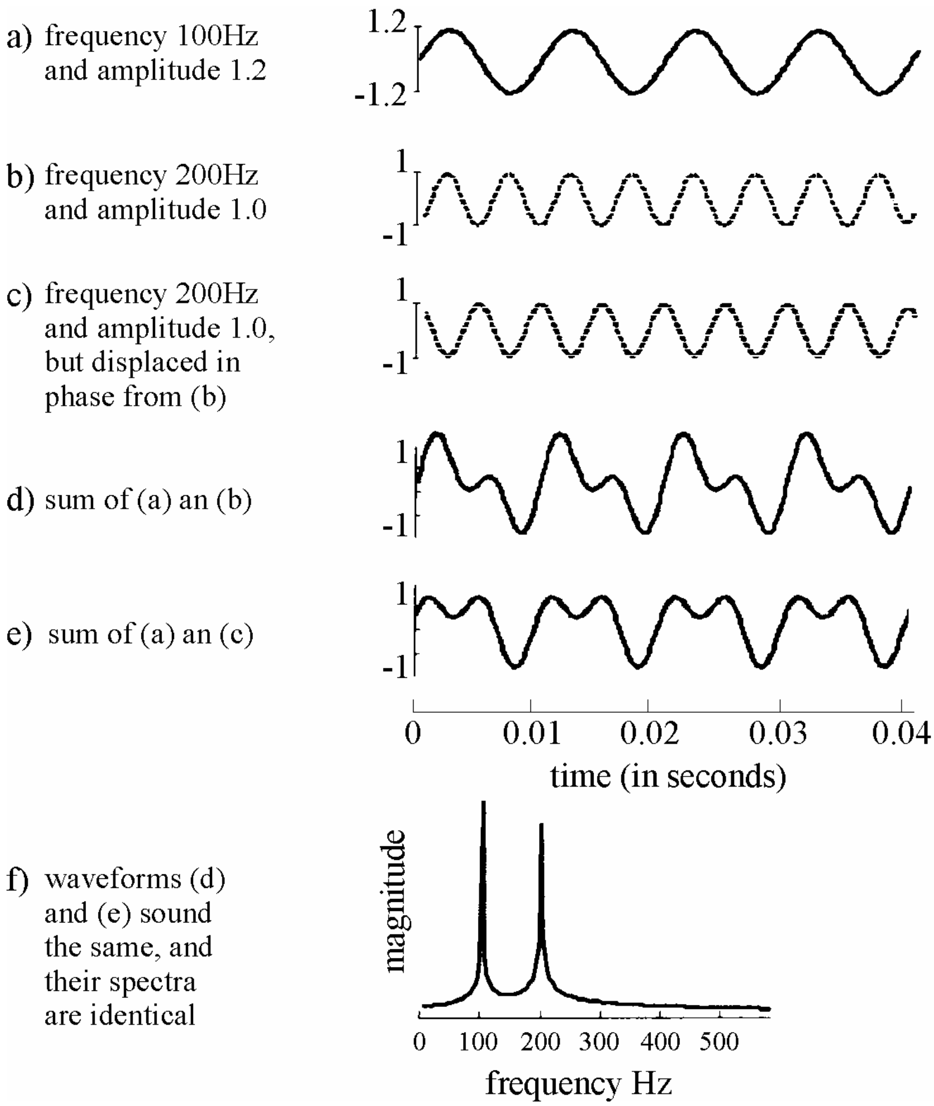
We are really interested in complex sound waves produced by our musical instruments and not in the pure sine waves. The pure sine waves are important only because the complex waves are made of them.

Just as a complex light wave is made of the rainbow spectrum of pure color waves, a complex sound wave is made of pure sine waves in various bass, midrange and treble frequencies. First could be analyzed by a prism, second by the Fourier analysis.

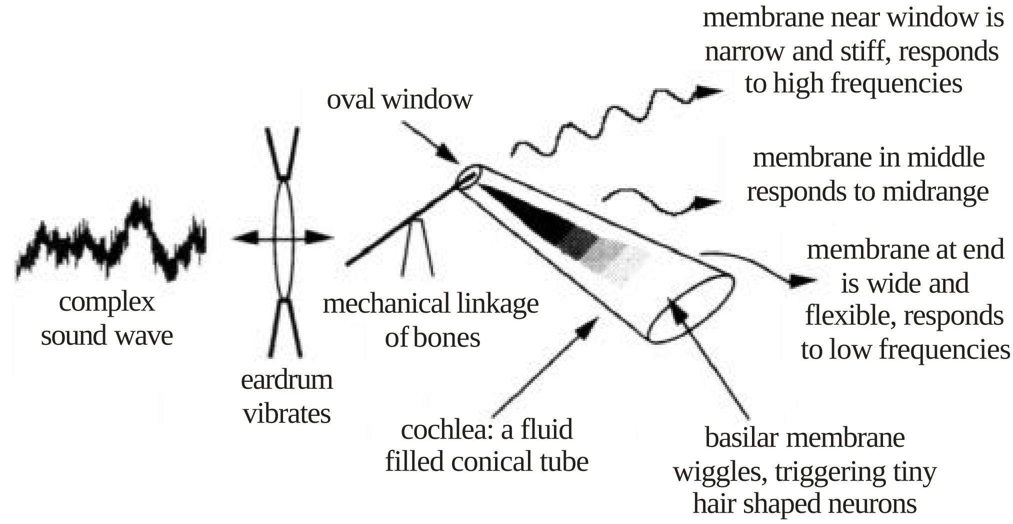




The Fourier analysis reduces a complex sound wave to its spectrum of frequencies. For example, the complex sound waves (d) and (e), which are (a) + (b) and (a) + (c) respectively, are both reduced to spectrum (f), which rediscovers the frequencies of the original sine waves.[8)]



Our auditory system is a biological spectrum analyzer doing the same. It transforms a sound wave into a frequency spectrum which has an auditory meaning. (G. Ohm was first to propose this idea in 1843.) This is explained in the following illustration.[9)]



The vibrations are transferred to the cochlea[10)] which is filled with fluid. The motion of the fluid rocks the membrane spread along the cochlea. The region nearest the oval window responds to high frequencies, while the far end responds to low frequencies. Tiny neurons sit on the membrane sending messages towards the brain when jostled.

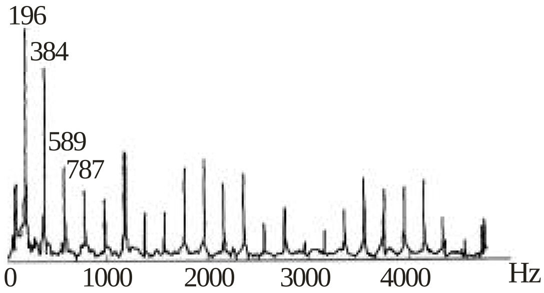
Thus the ear takes in a sound wave, like (d) or (e) above, and sends to the brain a representation of its spectrum, like (f) above. This representation has an auditory meaning.

6. HARMONIC AND NON-HARMONIC SOUNDS

As we said above, what really interests us is how to explain the dissonance of variety of intervals, when sounded by pairs of *complex* sound waves. These are sounds produced by our musical instruments.

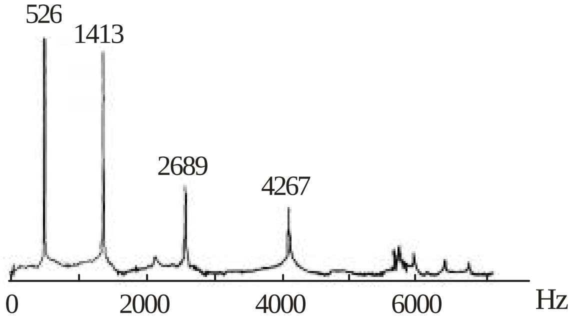
First of all there is a big difference between complex sounds that are *harmonic* and those that are *not harmonic*. We introduce these two kinds of sound with two examples.

A typical example of the harmonic sound is the sound of a guitar pluck. Here is its spectrum.



Notice that the spectrum consists of the fundamental at f = 196 Hz and of the near *integer* partials at $2f \approx 384$ Hz, $3f \approx 589$ Hz, $4f \approx 787$ Hz etc. Such a spectrum in which all the frequencies of vibration are integer multiples of some fundamental f is called *harmonic* and the corresponding sound is called *harmonic sound*. Since every partial repeats exactly within the period of the fundamental, harmonic sound waves are periodic.

A typical example of the non-harmonic sound is the sound of the strike of a metal bar. Here is its spectrum.



Notice that the spectrum consists of the fundamental at f = 526 Hz, and of the *non-integer* partials at $2.68\,f$ = 1413 Hz, $5.11f$ = 2689 Hz and $8.11\,f$ = 4267 Hz. Such a spectrum in which the frequencies of vibration are not the integer multiples of some fundamental f is called *non-harmonic* and the corresponding sound is called *non-harmonic sound*. Since at least some partials do not repeat exactly within the period of the fundamental, harmonic sound waves are not periodic.

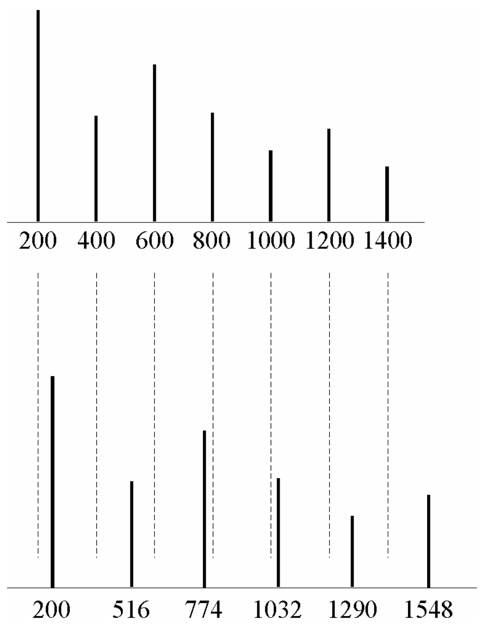
The guitar string and the metal bar are only two of many possible sound making devices. The harmonic vibrations of the string instruments are also characteristic of many other musical instruments. For example, when air oscillates in a wind instrument, its motion is constrained in the same way that the string is constrained by its fixed ends. At the closed end of the wind instrument the flow air must be zero, while at an open end the pressure must drop to zero. Thus all wind and string instruments have spectra which are harmonic. In contrast, most percussion instruments such as drums, marimbas, gongs etc. have non harmonic spectra.[11)]

7. THE TRUE THEORY FOR COMPLEX SOUNDS

Let us return to our main question. How to explain the dissonance of variety of intervals, when sounded by pairs of *complex* sound waves?

The Plomp-Levelt experiment gathered data only on perceptions of pure sine waves. A century before that, to explain the sensory dissonance of complex sounds, Helmholtz proposed the following procedure: *add up all of the dissonances, between all pairs of pure sine partials.*

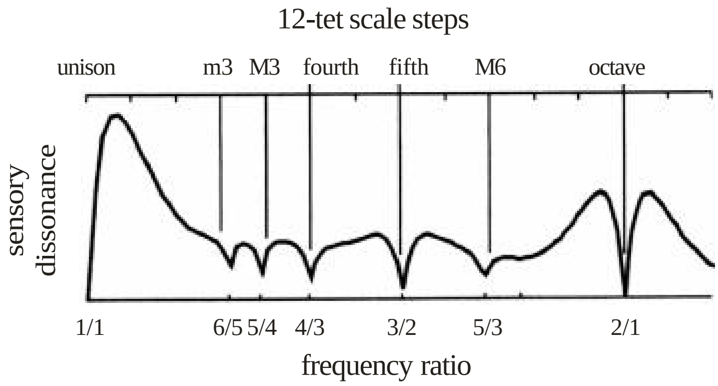
Notice that even if there is no beating interference of the fundamentals, there can be some beating interference of the other partials. Here is one example:



A harmonic sound at fundamental frequency f = 200 Hz is transposed to g = 258 Hz. When this interval is played simultaneously some of the partials interfere by beating rapidly, causing sensory dissonance.

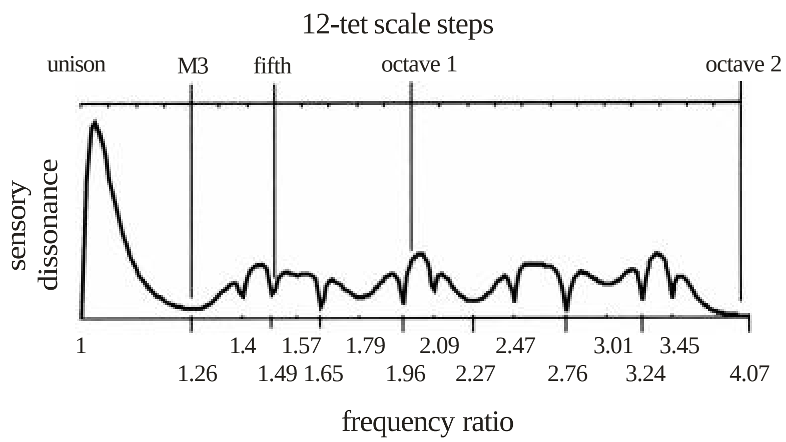
If we add up dissonances between all pairs of partials for all intervals we will get the dissonance curve for a given spectrum.

The dissonance curve for a harmonic spectrum with six partials at f, $2f$, $3f$, $4f$, $5f$ and $6f$ is shown in the following figure.[12)]



Notice, that minima of the dissonance curve coincide with many Pythagorean intervals, which are characterized by the law of small numbers. It is easy to prove that dissonance curves of harmonic spectra always have this property, and this is the *final explanation of the law of small numbers* for harmonic sounds.

The dissonance curve for a non harmonic spectrum of a metal bar with six partial at f, $2.76\,f$, $5.41\,f$, $8.94\,f$, $13.35\,f$ and $18.65\,f$ is shown in the following figure[13)].



Notice, that minima of the dissonance curve do not coincide with any of the Pythagorean intervals, which are characterized by the law of small numbers. It is easy to prove that dissonance curves of non-harmonic spectra always have this property, and this is the *final refutation of the law of small numbers* for non harmonic sounds.

8. CONCLUSION

We may conclude that the law of small numbers is just an epiphenomenon which is empirically irrelevant for our explanations as much as are the non-integer “laws” of slendro, pelog and other non-harmonic scales. The real source of our consonance discriminations is the phenomenon of beating, as hypothesized by Helmholtz and directly corroborated by Plomp-Levelt experiment[14)]. The beating is really something *unique* “out there” which we discriminate as dissonance “in here” and in this sense our discriminations are *objective* and not subjective. Furthermore, it is common to all musical traditions harmonic or not to discriminate between consonant and dissonant intervals in this way. It seems then that this is common to all humans, which means that our consonance discriminations are innate and not acquired. Hence, our sensory discriminations of consonant and dissonant musical intervals are *objective* and *innate* and this is corroborated by the western harmonic tradition as well as by the non-western non-harmonic traditions.

To be more specific we may say that sensory dissonance and consonance are functions of the interval *and the spectrum of the sound.* A scale and a spectrum are related if the dissonance curve for the spectrum has minima at the scale steps. Harmonic spectra of western musical instruments are related to western scales with many Pythagorean intervals. Non-harmonic spectra of different musical traditions are related to their scales.

And this is not the whole story. Nowadays musicians compose for very unusual sounds. In accordance with the previous explanations, their procedure is as follows:

(1) Choose a sound.
(2) Find the spectrum of the sound.
(3) Simplify the spectrum.
(4) Calculate the dissonance curve.
(5) Choose a set of intervals from the minima i.e. choose related scale.
(6) Create (synthesize) an instrument with the simplified spectrum that can play the sound at the chosen scale steps.
(7) Compose and play music.

We made the full circle. From music, to the first empirical laws, to their mathematical refinements and physical corroborations, and finely back to the music. To appreciate that you should listen to some music composed according to the procedure (1)–(7), which is the by product of this full circle history. The best starting point I can suggest is [S].

NOTES:

1) Note that it does not necessarily mean that intonation is not a human universal. (English language is not a human universal although language could be.)

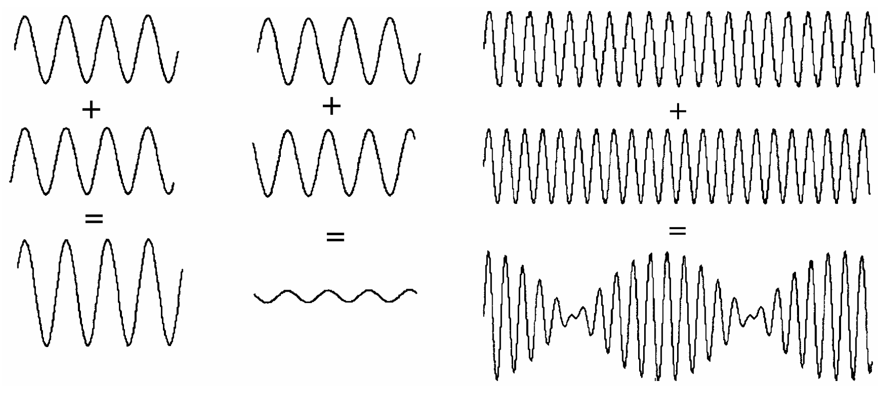
2) By innate I mean acquired by evolution at least in some respects. By acquired I mean acquired exclusively by culture.

3) Let me show you with some examples that our auditory discriminations can be of any of the four types. Our discrimination between loud and soft sound is innate and objective; between a string and a wind instrument it is acquired and objective; between an ugly and a beautiful piece of music it is acquired and subjective; between the mother and a foreign language it is innate and subjective (cf.[1)]).

4) My explanation follows [S].

5) The dissonance was defined as unpleasantness.
6) Helmholtz could make a relevant experiment only with complex sounds produced by then available instruments. The pure sounds, so easyavailable on computers these days, were not so easily available in Helmholtz's days.

7) The phenomenon of beating is caused by alternation of constructive and destructive interference. When two sine waves of exactly the same frequency are played together they sound just like a single wave, but the combination may be louder or softer then the original waves (see the left figure below). Since the waves on the left side have the same phase, the same starting point, their peaks and valleys line up exactly and the magnitude of the sum is greater then either are alone. This is constructive interference. When the waves on the right side of the left figure, which are out of phase, are added together the peaks of one line up with the valleys of the other and their sum is smaller then either alone. This is destructive interference.
What if the two sine waves differ slightly in frequency? The easiest way to picture this is to imagine that the two waves are at the same frequency, but that their relative phase slowly changes. When the phases are aligned they add constructively, while when out of phase they add destructively. The result is beating (see the right figure below).

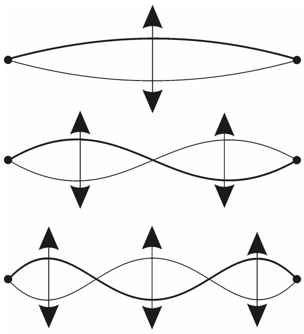


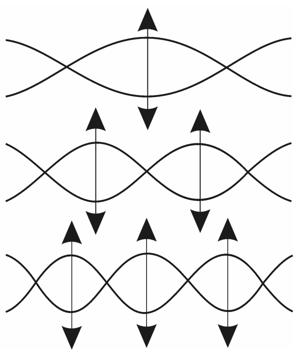
8) The figure is from [S] p.15.

9) Ibid. p. 16.

10) The cochlea is straightened out in the illustration. In reality it is curled up like a snail shell.

11) The spectrum of the string is harmonic because the string *is fixed at both ends*, and can only sustain oscillations that fit exactly into the length of the string (see the left figure below). It is possible to prove mathematically that for an ideal string, if the fundamental occurs at frequency f, the second partial must be at $2f$, the third at $3f$ etc.
The spectrum of the metal bar is non-harmonic because the bar is free at both ends. Hence, the movement of the struck bar is characterized by "bending modes" that specify how the bar will vibrate once it is set into motion (see the right figure below). It is possible to prove mathematically that for an ideal metal bar, if the fundamental occurs at frequency f, the second partial must be at $2.76f$, the third at $5.41f$, the fourth at $8.94f$ etc.





12) Ibid. p. 92. This is the figure that Helmholtz got as the result of his calculations.

13) Ibid. p. 107. Helmholtz did not make the calculations for non-harmonic spectra because he was focused exclusively on harmonic instruments.

14) It was indirectly corroborated by Helmholtz when he calculated that the minima of the dissonance curve for harmonic sounds (he was exclusively dealing with) correspond to Pythagorean intervals of the western harmonic tradition.

REFERENCES:

[B] Bernstein, L, *The unanswered Question*, Harvard University Press 1976.

[G] Galilei, G, *Two New Sciences* (translated by S. Drake), University of Wisconsin Press 1974.

[P] Perlman, M, *American gamelan in the garden of eden: intonation in a cross cultural encounter*, Musical Quarterly 78 (1994), 510-55.

[S] Sethares, W. A., *Tuning, Timbre, Spectrum, Scale*, Springer 1997.