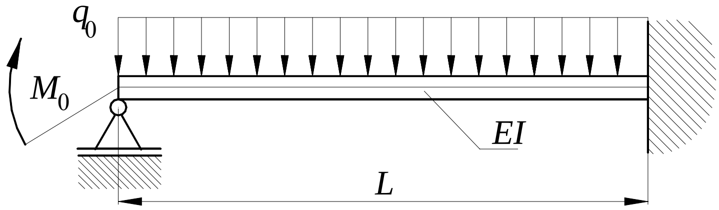
Rješenja 1. kolokvija 13.04.2010. iz kolegija
NUMERIČKE METODE U STROJARSTVU

1. Izvesti izraz za rješavanje problema štapa opterećnog na savijanje Galerkinovom metodom. Za nosač zadan i opterećen prema slici potrebno je primjenom Galerkinove metode odrediti raspodjelu pomaka i momenta savijanja. Za pretpostavljenu funkciju pomaka odabrati jednu od ponuđenih:

a) $\bar{w} = a\cos\left(\frac{\pi}{L}x\right)$, b) $\bar{w} = a\sin\left(\frac{\pi}{2L}x\right)$, c) $\bar{w} = a_1 x + a_2 x^2 + a_3 x^3$.

Zadano: L, $EI = konst.$, q_0, $M_0 = q_0 L^2$.



○Geometrijski rubni uvjeti:

$$\bar{w}(x=0)=0,\ \bar{w}(x=L)=0,\ \frac{\mathrm{d}\bar{w}}{\mathrm{d}x}(x=L)=0.$$

a) $\bar{w} = a\cos\left(\frac{\pi}{L}x\right)$: $0 = a\cos(0) \to a = 0$ ne zadovoljava,

b) $\bar{w} = a\sin\left(\frac{\pi}{2L}x\right)$: $0 = a\sin(0)$ zadovoljava, $0 = a\sin\left(\frac{\pi}{2L}L\right) = a\sin\left(\frac{\pi}{2}\right) \to a = 0$ ne zadovoljava,

c) $\bar{w} = a_1 x + a_2 x^2 + a_3 x^3 : 0 = a_1 0 + a_2 0^2 + a_3 0^3$ zadovoljava, $0 = a_1 L + a_2 L^2 + a_3 L^3 \to a_1 = -a_2 L - a_3 L^2$,

$0 = a_1 + 2a_2 L + 3a_3 L^2 \to a_1 = -2a_2 L - 3a_3 L^2$. Iz ova dva dodatna uvjeta može se zapisati

$a_1 = -a_2 L - a_3 L^2 = -2a_2 L - 3a_3 L^2 \to a_2 = -2a_3 L$,

$a_1 = -2(-2a_3 L)L - 3a_3 L^2 = a_3 L^2$ pa ovi uvjeti vraćeni u početni izraz daju

$\bar{w} = a_3 L^2 x - a_3 L x^2 + a_3 x^3 = a_3\left(x^3 - 2Lx^2 + L^2 x\right)$.

Odabrana funkcija je pod c)!

$$\bar{w} = a_3\left(x^3 - 2Lx^2 + L^2 x\right).$$

Izvod jednadžbe Galerkinove metode za riješavanje problema savijanja tankih štapova je u knjizi Uvod u numeričke metode u strojarstvu, Jurica Sorić, stranica 61. Jednadžba glasi:

$$\int_0^l \left(EI_y \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2}\frac{\mathrm{d}^2 f_i}{\mathrm{d}x^2} - q_z f_i \right)\mathrm{d}x + \left[f_i EI_y \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}x^3} \right]_0^l - \left[\frac{\mathrm{d}f_i}{\mathrm{d}x} EI_y \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2} \right]_0^l = 0.$$

Težinska funkcija

$f_1 = x^3 - 2Lx^2 + L^2 x$.

Prirodni rubni uvjeti:

$M(x=0) = -M_0$.

$\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} = a_3\left(3x^2 - 4Lx + L^2\right)$, $\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2} = a_3(6x - 4L)$.

$\frac{\mathrm{d}f_1}{\mathrm{d}x} = 3x^2 - 4Lx + L^2$, $\frac{\mathrm{d}^2 f_1}{\mathrm{d}x^2} = 6x - 4L$.

$$EI_y\int_0^L a_3(6x-4L)(6x-4L)\,\mathrm{d}x - q_0\int_0^L\left(x^3 - 2Lx^2 + L^2 x\right)\mathrm{d}x - \left[\left(3x^2 - 4Lx + L^2\right)(-M_b)\right]_{x=0} = 0.$$

$$EI_y a_3 \int_0^L \left(36x^2 - 48xL + 16L^2\right) \mathrm{d}x - q_0 \int_0^L \left(x^3 - 2Lx^2 + L^2 x\right) \mathrm{d}x - \left[\left(L^2\right)\left(-\left(-M_0\right)\right)\right]_{x=0} = 0 .$$

$$EI_y a_3 4L^3 - q_0 \frac{L^4}{12} - M_0 L^2 = 0 .$$

$$a_3 = \frac{13}{48} \frac{q_0 L}{EI_y} .$$

$$\overline{w} = \frac{13}{48} \frac{q_0 L}{EI_y} \left(x^3 - 2Lx^2 + L^2 x\right).$$

Moment savijanja

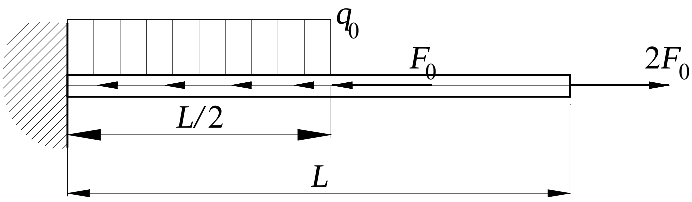
$$\overline{M}_y = -EI \frac{\mathrm{d}^2 \overline{w}}{\mathrm{d}x^2}$$

$$\overline{M}_y = \frac{13}{48} q_0 L \left(-6x + 4L\right)$$

2. Za štap zadan i opterećen prema slici potrebno je primjenom Rayleigh-Ritzove metode odrediti raspodjelu pomaka i uzdužne sile. Za pretpostavljenu funkciju pomaka uzeti:

$\overline{u} = a_1 x + a_2 x^2$.

Zadano: $L, AE =$ konst., q_0, $F_0 = q_0 L$.



○Geometrijski rubni uvjet:

$$\overline{u}(x=0) = 0 .$$

Pretpostavljena funkcija pomaka zadovoljava geometrijske rubne uvjete: $0 = a_1 0 + a_2 0^2$.

Funkcional za zadani štap je

$$\Pi = \frac{1}{2} AE \int_0^L \left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}x}\right)^2 dx - \int_0^{\frac{L}{2}} q_x \overline{u} dx - \left[-F_0 \overline{u}\right]_{\frac{L}{2}} - \left[2F_0 \overline{u}\right]_L .$$

$$\Pi = \frac{1}{2} AE \int_0^L \left(a_1 + 2a_2 x\right)^2 dx - \int_0^{\frac{L}{2}} -q_0 \left(a_1 x + a_2 x^2\right) dx - \left[-F_0 \left(a_1 x + a_2 x^2\right)\right]_{\frac{L}{2}} - \left[2F_0 \left(a_1 x + a_2 x^2\right)\right]_L .$$

$$\Pi = \frac{1}{2} AE \left(a^2{}_1 L + 2a_1 a_2 L^2 + \frac{4a^2{}_2 L^3}{3}\right) + q_0 \left(\frac{a_1 L^2}{8} + \frac{a_2 L^3}{24}\right) + F_0 \left(a_1 \frac{L}{2} + a_2 \frac{L^2}{4}\right) - 2F_0 \left(a_1 L + a_2 L^2\right) .$$

$$\Pi = \frac{1}{2} AE \left(a^2{}_1 L + 2a_1 a_2 L^2 + \frac{4a^2{}_2 L^3}{3}\right) + q_0 \left(-\frac{11a_1 L^2}{8} - \frac{41a_2 L^3}{24}\right).$$

$$\frac{\partial \Pi}{\partial a_1} = 0, \frac{\partial \Pi}{\partial a_2} = 0$$

$$\frac{\partial \Pi}{\partial a_1} = AE\left(a_1 L + a_2 L^2\right) + q_0\left(-\frac{11L^2}{8}\right) = 0,$$

$$\frac{\partial \Pi}{\partial a_2} = AE\left(a_1 L^2 + \frac{4a_2 L^3}{3}\right) + q_0\left(-\frac{41L^3}{24}\right) = 0.$$

$$a_1 L + a_2 L^2 = \frac{11 q_0 L^2}{8AE}.$$

$$a_1 L^2 + \frac{4a_2 L^3}{3} = \frac{41 q_0 L^3}{24AE}$$

$$a_1 = \frac{3}{8}\frac{q_0 L}{AE},$$

$$a_2 = \frac{q_0}{AE}.$$

$$\bar{u} = \frac{3}{8}\frac{q_0 L}{AE}x + \frac{q_0}{AE}x^2.$$

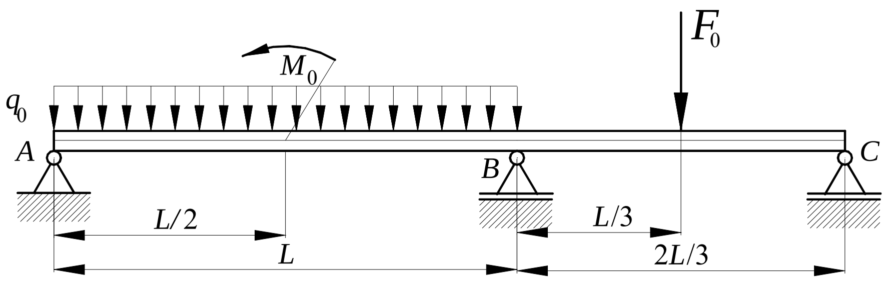
Uzdužna sila:

$$\bar{N}_x = AE\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}.$$

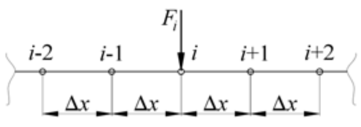
$$\bar{N}_x = \frac{3}{8}q_0 L + q_0 x.$$

3. Za nosač zadan i opterećen prema slici potrebno je primjenom metode konačnih razlika izvesti sustav jednadžbi u matričnom obliku za određivanje nepozantih progiba. Provesti diskretizaciju s $n = 6$ jednako razmaknutih čvorova. Izvesti izraz za rubni uvjet u osloncu *A*. Izvesti izraz za izračunavanje reakcije u osloncu *C*.

Zadano: $L, EI = \text{konst.}, q_0, F_0 = q_0 L, M_0 = q_0 L^2$.



Zadatku mogu korisno poslužiti sljedeće relacije:



$$w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2} = \frac{(\Delta x)^3}{EI}F_i,$$

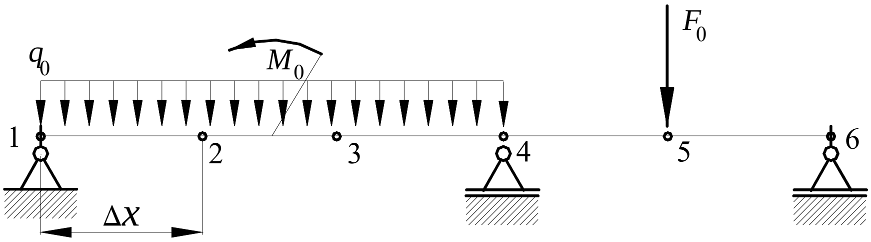
○Geometrijski rubni uvjeti:

$$\bar{w}(x=0) = 0, \bar{w}(x=L) = 0, \bar{w}(x=\frac{5L}{3}) = 0.$$

○Prirodni rubni uvjeti:

$$M(x=0) = M_1 = 0, M(x=\frac{5L}{3}) = M_6 = 0.$$

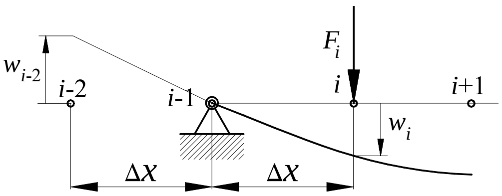
Proračunski model:



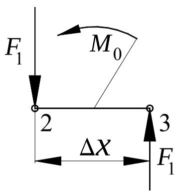
$$\Delta x = \frac{L}{3}$$

Diferencijske jednadžbe po čvorovima:

Čvor 2:



$$w_{i-2} = -w_i, \quad 5w_i - 4w_{i+1} + w_{i+2} = \frac{(\Delta x)^3}{EI} F_i \;.$$

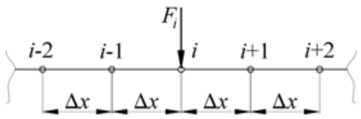


$$M_0 = F_1 \Delta x \rightarrow F_1 = \frac{M_0}{\Delta x} = \frac{q_0 9 (\Delta x)^2}{\Delta x} = 9 q_0 \Delta x \;.$$

$$5w_2 - 4w_3 + w_4 = \frac{(\Delta x)^3}{EI} \left(q_0 \Delta x + 9 q_0 \Delta x \right) = 10 \frac{q_0 (\Delta x)^4}{EI}$$

$$5w_2 - 4w_3 = 10 \frac{q_0 (\Delta x)^4}{EI} \;.$$

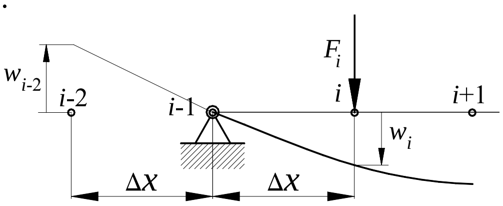
Čvor 3:



$$w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2} = \frac{(\Delta x)^3}{EI} F_i \,,$$

$$-4w_2 + 6w_3 + w_5 = \frac{(\Delta x)^3}{EI} \left(q_0 \Delta x - 9 q_0 \Delta x \right) = -8 \frac{q_0 (\Delta x)^4}{EI} \;.$$

Čvor 5:



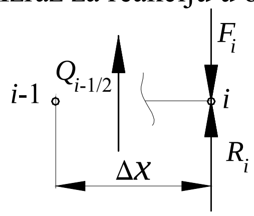
$$w_{i-2} = -w_i, \quad 5w_i - 4w_{i+1} + w_{i+2} = \frac{(\Delta x)^3}{EI} F_i \;.$$

$$5w_5 - 4w_4 + w_3 = \frac{(\Delta x)^3}{EI} \left(3 q_0 \Delta x \right) = 3 \frac{q_0 (\Delta x)^4}{EI} \;.$$

$$5w_5 + w_3 = 3 \frac{q_0 (\Delta x)^4}{EI} \;.$$

$$\begin{bmatrix} 5 & -4 & 0 \\ -4 & 6 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \\ w_5 \end{bmatrix} = \frac{q_0 (\Delta x)^4}{EI} \begin{bmatrix} 10 \\ -8 \\ 3 \end{bmatrix} .$$

Izraz za reakciju u osloncu C:



$$\sum F_z = 0 : F_i - R_i - Q_{i-\frac{1}{2}} = 0 \rightarrow R_i = F_i - Q_{i-\frac{1}{2}} .$$

$$Q_{i-\frac{1}{2}} = \left.\frac{\mathrm{d}M}{\mathrm{d}x}\right|_{i-\frac{1}{2}} ; \left.\frac{\mathrm{d}M}{\mathrm{d}x}\right|_{i-\frac{1}{2}} \approx \frac{M_i - M_{i-1}}{\Delta x} ; M_i = 0, M_{i-1} \approx -EI \frac{w_i - 2w_{i-1} + w_{i-2}}{\left(\Delta x\right)^2} .$$

$$Q_{i-\frac{1}{2}} \approx \frac{0 - EI \dfrac{-w_i + 2w_{i-1} - w_{i-2}}{\left(\Delta x\right)^2}}{\Delta x} = EI \frac{w_i - 2w_{i-1} + w_{i-2}}{\left(\Delta x\right)^3} .$$

$$R_i = F_i - \frac{w_i - 2w_{i-1} + w_{i-2}}{\left(\Delta x\right)^3} = F_i + EI \frac{2w_5}{\left(\Delta x\right)^3} .$$