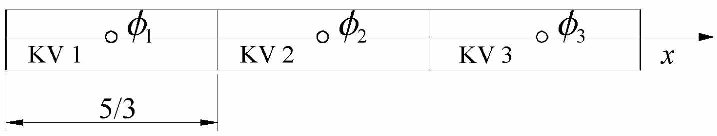
RJEŠENJA 2. KOLOKVIJA
NUMERIČKE METODE U STROJARSTVU

1. Potrebno je izvesti sustav jednadžbi za rješavanje jednodimenzijske Poissonove diferencijalne jednadžbe $5\dfrac{\mathrm{d}^2\phi}{\mathrm{d}x^2}=10x$ zadane na području $0\le x\le 5$. Problem diskretizirati s tri volumena.

Zadani rubni uvjeti: $\phi(5)=40,\ 5\dfrac{\mathrm{d}\phi}{\mathrm{d}x}(0)=100$.

Proračunski model:



Integrirjući jednadžbu $5\dfrac{\mathrm{d}^2\phi}{\mathrm{d}x^2}=10x$ po konačnom volumenu dobivamo

$$\int_V 5\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)\mathrm{d}V=\int_V 10x\,\mathrm{d}V \ .$$

$$\int_V 5\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)\mathrm{d}V=5\int_S\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)\mathrm{n}_x\mathrm{d}S \ .$$

$$5\oint_S\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)n_x\mathrm{d}S=5\sum_c\int_{S_c}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)n_x^c\mathrm{d}S_c \ .$$

$$5\sum_c\int_{S_c}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)n_x^c\mathrm{d}S_c=5\sum_c\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_c n_x^c\Delta S_c \ .$$

$$5\sum_c\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_c n_x^c\Delta S_c=10x_P\Delta V \ .$$

$c=e,w$ su sredine stranica volumena, n_x^c komponenta normale po osi x na sredini stranice c, ΔS_c ploština stranice c, x_P koordinata središta volumena, ΔV veličina volumena.

$$5\left\{\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e-\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w\right\}=10x_P\Delta x \ .$$

KV 1:

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e\approx\frac{\phi_2-\phi_1}{5/3};5\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w=100\rightarrow\left\{5\frac{\phi_2-\phi_1}{5/3}-100\right\}=10\frac{5}{6}\frac{5}{3}.$$

$$-3\phi_1+3\phi_2=\frac{2050}{18} \ .$$

KV 2:

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e\approx\frac{\phi_3-\phi_2}{5/3};\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w\approx\frac{\phi_2-\phi_1}{5/3}\rightarrow 5\left\{\frac{\phi_3-\phi_2}{5/3}-\frac{\phi_2-\phi_1}{5/3}\right\}=10\frac{5}{2}\frac{5}{3}.$$

$$3\phi_1-6\phi_2+3\phi_3=\frac{750}{18} \ .$$

KV 3:

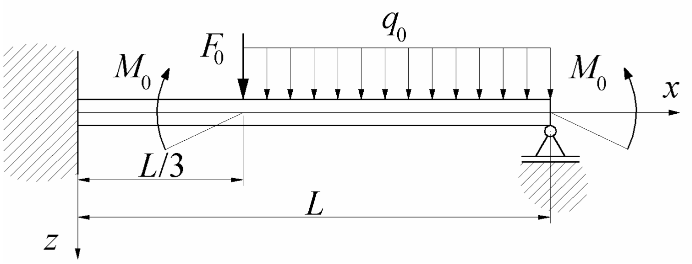
$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e\approx\frac{\phi_e^{0}-\phi_3}{5/6};\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w\approx\frac{\phi_3-\phi_2}{5/3}\rightarrow 5\left\{\frac{40-\phi_3}{5/6}-\frac{\phi_3-\phi_2}{5/3}\right\}=10\frac{25}{6}\frac{5}{3}.$$

$$3\phi_2-9\phi_3=\frac{-3070}{18} \ .$$

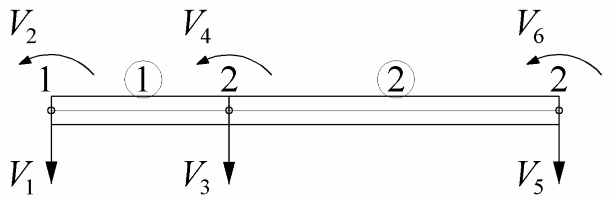
$$\begin{bmatrix} -3 & 3 & 0 \\ 3 & -6 & 3 \\ 0 & 3 & -9 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 2050 \\ 750 \\ -3070 \end{bmatrix}.$$

3. Za nosač zadan i opterećen prema slici potrebno je odrediti vektor čvornih opterećenja prikazan u globalnim stupnjevima slobode pomoću metode konačnih elemenata. Problem diskretizirati s dva osnovna gredna elementa. Prikazati izračunavanje raspodjele momenta savijanja za element koji uključuje uklještenje, bez izračunavanja stupnjeva slobode.

Zadano: $L, q_0, F_0 = q_0 L, M_0 = q_0 L^2, EI =$ konst.



$$\mathbf{N} = \left[\left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) \quad \left(-x + \frac{2x^2}{l} - \frac{x^3}{l^2}\right) \quad \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) \quad \left(\frac{x^2}{l} - \frac{x^3}{l^2}\right)\right], \; \mathbf{F}_s^T = q_0 L\begin{bmatrix} \frac{1}{3} & -\frac{L}{27} & \frac{1}{3} & \frac{L}{27} \end{bmatrix}.$$



$$\mathbf{V}^T = \begin{bmatrix} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{bmatrix}.$$

$$\mathbf{R}^T = \begin{bmatrix} R_1 & R_2 & R_3 & R_4 & R_5 & R_6 \end{bmatrix}.$$

$$\left(\mathbf{F}_s^2\right)^T = q_0 L\begin{bmatrix} \frac{1}{3} & -\frac{L}{27} & \frac{1}{3} & \frac{L}{27} \end{bmatrix}.$$

$$\left(\mathbf{R}_s^2\right)^T = q_0 L\begin{bmatrix} 0 & 0 & \frac{1}{3} & -\frac{L}{27} & \frac{1}{3} & \frac{L}{27} \end{bmatrix}.$$

$$\left(\mathbf{F}\right)^T = q_0 L\begin{bmatrix} 0 & 0 & 1 & -L & 0 & L \end{bmatrix}.$$

$$\left(\mathbf{R}\right)^T = q_0 L\begin{bmatrix} 0 & 0 & \frac{1}{3}+1 & -\frac{L}{27}-L & \frac{1}{3} & \frac{L}{27}+L \end{bmatrix} = q_0 L\begin{bmatrix} 0 & 0 & \frac{4}{3} & -\frac{28L}{27} & \frac{1}{3} & \frac{28L}{27} \end{bmatrix}.$$

Raspodjela momenta savijanja po elementu 1:

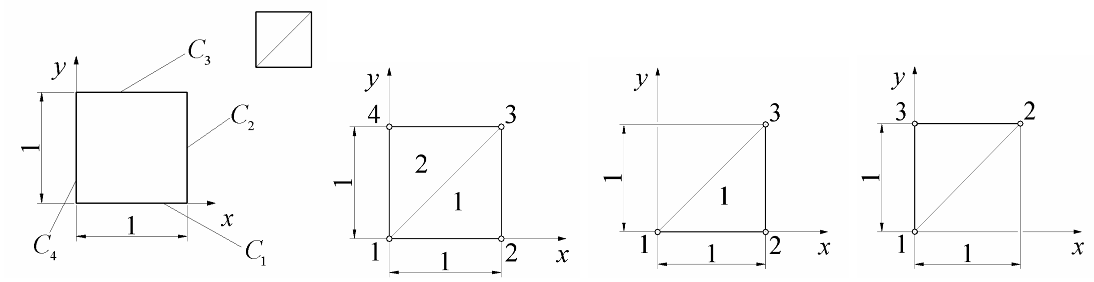
$$M_y^{\,1} = -EI\frac{\mathrm{d}^2 w^1}{\mathrm{d}x^2} = -EI\begin{bmatrix} \frac{\mathrm{d}^2 N_1}{\mathrm{d}x^2} & \frac{\mathrm{d}^2 N_2}{\mathrm{d}x^2} & \frac{\mathrm{d}^2 N_3}{\mathrm{d}x^2} & \frac{\mathrm{d}^2 N_4}{\mathrm{d}x^2} \end{bmatrix}\begin{bmatrix} v_1^{\,1} \\ v_2^{\,1} \\ v_3^{\,1} \\ v_4^{\,1} \end{bmatrix}.$$

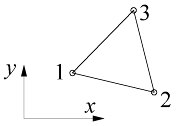
$$M_y^{\,1} = -EI\left[\left(\frac{-6}{l^2} + \frac{12x}{l^3}\right) \quad \left(\frac{4}{l} - \frac{6x}{l^2}\right) \quad \left(\frac{6}{l^2} - \frac{12x}{l^3}\right) \quad \left(\frac{2}{l} - \frac{6x}{l^2}\right)\right]\begin{bmatrix} 0 \\ 0 \\ V_3 \\ V_4 \end{bmatrix}.$$

4. Zadan je problem provođenja topline s konstantnim toplinskim izvorom q. Za diskretizaciju s dva trokutna elementa prema slici izračunati temperature u čvorovima. Za prikazane konačne elemente matrice krutosti su

$$\mathbf{k}^1 = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix},\ \mathbf{k}^2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$ Zadano: $\rho = 1\ \text{kg/m}^3$, $\lambda = 16\ \text{W/mK}$, $q = 1\ \text{W/kg}$.

Rubni uvjeti su: $C_1 : \lambda \frac{\partial T}{\partial y} = 0;\ C_2 : \lambda \frac{\partial T}{\partial x} = 96;\ C_3 : T = 3x^2 + 5;\ C_4 : \lambda \frac{\partial T}{\partial x} = 0.$



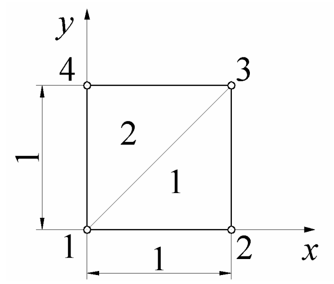


$$\mathbf{k} = \frac{1}{4A}\begin{bmatrix} \beta_1^2 + \gamma_1^2 & \beta_1\beta_2 + \gamma_1\gamma_2 & \beta_1\beta_3 + \gamma_1\gamma_3 \\ \beta_2\beta_1 + \gamma_2\gamma_1 & \beta_2^2 + \gamma_2^2 & \beta_2\beta_3 + \gamma_2\gamma_3 \\ \beta_3\beta_1 + \gamma_3\gamma_1 & \beta_3\beta_2 + \gamma_3\gamma_2 & \beta_3^2 + \gamma_3^2 \end{bmatrix},\ \int_A N_1^i\, N_2^j\, N_3^k\, dA = \frac{i!\,j!\,k!}{(i+j+k+2)!}2A$$

$$\begin{aligned} &\alpha_1 = x_2 y_3 - x_3 y_2, \quad \beta_1 = y_2 - y_3, \quad \gamma_1 = x_3 - x_2, \\ &\alpha_2 = x_3 y_1 - x_1 y_3, \quad ,\beta_2 = y_3 - y_1, \quad ,\ \gamma_2 = x_1 - x_3, \quad ,N_i = \frac{1}{2A}\left(\alpha_i + x\beta_i + y\gamma_i\right),\ i = 1,3\,. \\ &\alpha_3 = x_1 y_2 - x_2 y_1 \quad \beta_3 = y_1 - y_2 \quad \gamma_3 = x_2 - x_1 \end{aligned}$$

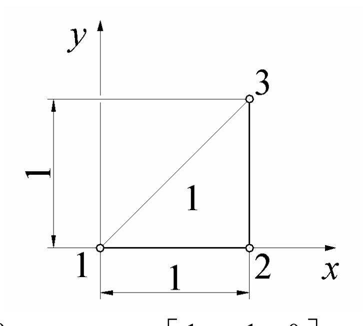
$$-\frac{\partial^2 T}{\partial x_i \partial x_i} = \frac{\rho q}{\lambda}.$$

Proračunski model:

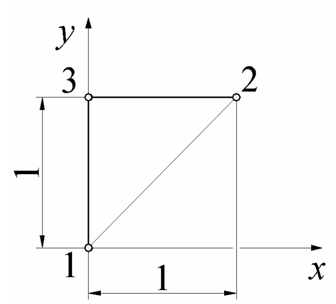


$\mathbf{T}^T = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix}.$

KE 1:



$$\begin{aligned} &\beta_1 = -1, \quad \gamma_1 = 0, \\ &\beta_2 = 1, \quad \gamma_2 = -1, \quad \mathbf{k}^1 = \frac{1}{4\cdot 0{,}5}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \\ &\beta_3 = 0 \quad \gamma_3 = 1 \end{aligned}$$



$$\beta_1 = 0,\quad \gamma_1 = -1,$$
$$\beta_2 = 1,\quad \gamma_2 = 0,\quad \mathbf{k}^2 = \frac{1}{4\cdot 0,5}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$
$$\beta_3 = -1\quad \gamma_3 = 1$$

Globalne čvorne varijable		1	2	3	4
Elementi, lokalne čv. var.	1	1	2	3	-
	2	1	-	2	3

$$\mathbf{K}^1 = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\mathbf{K}^2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

$$\mathbf{F}_S^1 = \int_A \mathbf{N}^T f \,\mathrm{d}A = \int_A \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\rho q}{\lambda}\,\mathrm{d}A\,.\quad \int_A N_1^i\, N_2^j\, N_3^k\, dA = \frac{i!\,j!k!}{(i+j+k+2)!}2A\,.$$

$$\int_A N_1 \;\; dA = \frac{1!}{(1+2)!}2\cdot 0.5 = \frac{1}{6}\,.$$

$$\int_A N_2 \;\; dA = \frac{1!}{(1+2)!}2\cdot 0.5 = \frac{1}{6}\,.$$

$$\int_A N_3 \;\; dA = \frac{1!}{(1+2)!}2\cdot 0.5 = \frac{1}{6}\,.$$

$$\mathbf{F}_S^1 = \frac{1}{16}\frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.\quad \mathbf{F}_S^2 = \frac{1}{16}\frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\mathbf{R}_S^1 = \frac{1}{16}\frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.\quad \mathbf{R}_S^2 = \frac{1}{16}\frac{1}{6}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$\mathbf{R}_S = \mathbf{R}_S^1 + \mathbf{R}_S^2 = \frac{1}{16}\frac{1}{6}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

$$\mathbf{K} = \mathbf{K}^1 + \mathbf{K}^2 = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

Prirodni rubni uvjeti

$$C_2 : \mathbf{F}_b^1 = \int_l \mathbf{N}^T b_c \mathrm{d}l \,.$$

$$\mathbf{F}_b^1 = \int_0^1 \begin{bmatrix} 0 \\ 1-y \\ y \end{bmatrix} 6 \; \mathrm{d}y = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}.$$

$$\mathbf{R}_b^1 = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}.$$

$$\mathbf{R} = \mathbf{R}_S + \mathbf{R}_b^1 = \frac{1}{96}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 2 \\ 289 \\ 290 \\ 1 \end{bmatrix}.$$

Osnovni rubni uvjeti

$$T_3 = 8, T_4 = 5 \,.$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 2 \\ 289 \\ 290 \\ 1 \end{bmatrix},$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 2 \\ 289 \\ 290 \\ 1 \end{bmatrix} - 8\frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix},$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 2 \\ 289 \\ 290 \\ 1 \end{bmatrix} - 8\frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} - 5\frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 2+240 \\ 289+384 \\ 290 \\ 1 \end{bmatrix}.$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 242 \\ 673 \end{bmatrix}.$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 8,0347 \\ 11,0278 \end{bmatrix}.$$