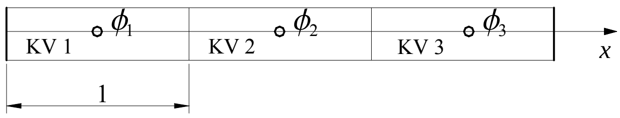
RJEŠENJA 2. KOLOKVIJA
NUMERIČKE METODE U STROJARSTVU

1. . Potrebno je izvesti sustav jednadžbi za rješavanje jednodimenzijske Poissonove diferencijalne jednadžbe $\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}=-240x^2$ zadane na području $0 \le x \le 3$ pomoću metode konačnih volumena. Problem diskretizirati s tri volumena.

Zadani rubni uvjeti: $\phi(3)=-1600,\ \frac{\mathrm{d}\phi}{\mathrm{d}x}(0)=7$.

Proračunski model:



Integrirjući jednadžbu $\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}=-240x^2$ po konačnom volumenu dobivamo

$$\int_V \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\mathrm{d}V=\int_V -240x^2\mathrm{d}V\,,$$

$$\int_V \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)\mathrm{d}V=\int_V -240x^2\mathrm{d}V$$

$$\int_V \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)\mathrm{d}V=\int_S \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)\mathrm{n}_x\mathrm{d}S\,.$$

$$\oint_S \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)n_x\mathrm{d}S=\sum_c\int_{S_c}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)n_x^c\mathrm{d}S_c\,.$$

$$\sum_c\int_{S_c}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)n_x^c\mathrm{d}S_c=\sum_c\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_c n_x^c\Delta S_c\,.$$

$$\sum_c\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_c n_x^c\Delta S_c=-240\left(x_P\right)^2\Delta V\ \ .$$

$c=e,w$ su sredine stranica volumena, n_x^c komponenta normale po osi x na sredini stranice c, ΔS_c ploština stranice c , x_P koordinata središta volumena, ΔV veličina volumena.

$$\left\{\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e-\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w\right\}=-240\left(x_P\right)^2\Delta x\ \ .$$

KV 1:

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e\approx\frac{\phi_2-\phi_1}{1};\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w=7\rightarrow\left\{\phi_2-\phi_1-7\right\}=-240\left(0,5\right)^2.$$

$$-\phi_1+\phi_2=-53$$

.

KV 2:

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e\approx\frac{\phi_3-\phi_2}{1};\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w\approx\frac{\phi_2-\phi_1}{1}\rightarrow\left\{\phi_3-\phi_2-\left(\phi_2-\phi_1\right)\right\}=-240\left(1,5\right)^2.$$

$$\phi_1-2\phi_2+\phi_3=-540$$

.

KV 3:

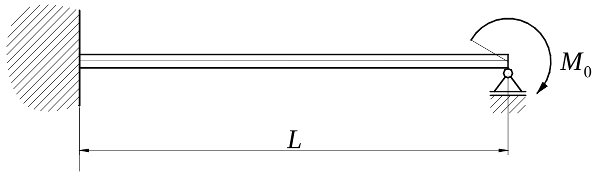
$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_e\approx\frac{\phi_e^{\ 0}-\phi_3}{1/2};\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_w\approx\frac{\phi_3-\phi_2}{1}\rightarrow\left\{\frac{-1600-\phi_3}{1/2}-\left(\phi_3-\phi_2\right)\right\}=-240\left(2,5\right)^2.$$

$$\phi_2 - 3\phi_3 = 1700$$

.

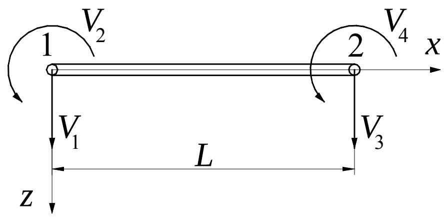
$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} -53 \\ -540 \\ 1700 \end{bmatrix}.$$

3. Za nosač zadan i opterećen prema slici potrebno je odrediti raspodjelu momenta savijanja pomoću metode konačnih elemenata. Problem diskretizirati osnovnim grednim elementom.
Zadano: $L, M_0, EI =$ konst.



$$\mathbf{N} = \left[\left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) \quad \left(-x + \frac{2x^2}{l} - \frac{x^3}{l^2}\right) \quad \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) \quad \left(\frac{x^2}{l} - \frac{x^3}{l^2}\right) \right], \ \mathbf{k} = EI \begin{bmatrix} \frac{12}{l^3} & -\frac{6}{l^2} & -\frac{12}{l^3} & -\frac{6}{l^2} \\ -\frac{6}{l^2} & \frac{4}{l} & \frac{6}{l^2} & \frac{2}{l} \\ -\frac{12}{l^3} & \frac{6}{l^2} & \frac{12}{l^3} & \frac{6}{l^2} \\ -\frac{6}{l^2} & \frac{2}{l} & \frac{6}{l^2} & \frac{4}{l} \end{bmatrix}.$$

Proračunski model



$$\mathbf{V}^T = [V_1 \quad V_2 \quad V_3 \quad V_4].$$
$$(\mathbf{R})^T = M_0 [0 \quad 0 \quad 0 \quad -1].$$

Jednadžba konačnog elementa

$$EI \begin{bmatrix} \frac{12}{L^3} & -\frac{6}{L^2} & -\frac{12}{L^3} & -\frac{6}{L^2} \\ -\frac{6}{L^2} & \frac{4}{L} & \frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & \frac{6}{L^2} & \frac{12}{L^3} & \frac{6}{L^2} \\ -\frac{6}{L^2} & \frac{2}{L} & \frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = M_0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Rubni uvjeti pomaka:
$V_1 = 0, V_2 = 0, V_3 = 0$.
Nakon uvođenja rubnih uvjeta pomaka ostaje jednadžba

$$EI \frac{4}{L} V_4 = -M_0 .$$

$$V_4 = -\frac{M_0 L}{4EI}.$$

Raspodjela momenta savijanja po elementu 1:

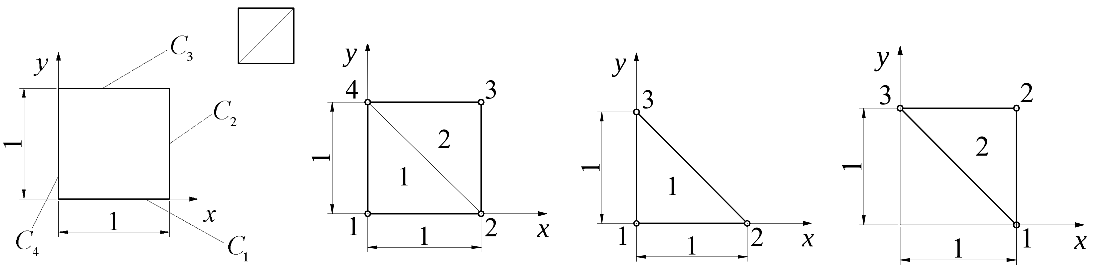
$$M_y = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -EI\begin{bmatrix}\frac{\mathrm{d}^2 N_1}{\mathrm{d}x^2} & \frac{\mathrm{d}^2 N_2}{\mathrm{d}x^2} & \frac{\mathrm{d}^2 N_3}{\mathrm{d}x^2} & \frac{\mathrm{d}^2 N_4}{\mathrm{d}x^2}\end{bmatrix}\begin{bmatrix}V_1\\V_2\\V_3\\V_4\end{bmatrix}.$$

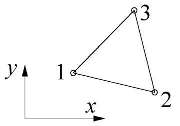
$$M_y = -EI\begin{bmatrix}\left(\frac{-6}{L^2}+\frac{12x}{L^3}\right) & \left(\frac{4}{L}-\frac{6x}{L^2}\right) & \left(\frac{6}{L^2}-\frac{12x}{L^3}\right) & \left(\frac{2}{L}-\frac{6x}{L^2}\right)\end{bmatrix}\begin{bmatrix}0\\0\\0\\-\frac{M_0 L}{4EI}\end{bmatrix} = M_0\left(\frac{1}{2}-\frac{3}{2}\frac{x}{L}\right).$$

4. Zadan je problem provođenja topline s konstantnim toplinskim izvorom q. Za diskretizaciju s dva trokutna elementa prema slici izračunati temperaturu u sredini ploče. Za prikazane konačne elemente matrice koeficijenata su

$\mathbf{k}^1 = \frac{1}{2}\begin{bmatrix}2 & -1 & -1\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$, $\mathbf{k}^2 = \frac{1}{2}\begin{bmatrix}1 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}$. Zadano: $\rho = 1\ \mathrm{kg/m^3},\ \alpha = 16\ \mathrm{W/mK},\ q = 1\ \mathrm{W/kg}$.

Rubni uvjeti su: $C_1 : \alpha\frac{\partial T}{\partial y} = 0;\ C_2 : \alpha\frac{\partial T}{\partial x} = 96;\ C_3 : T = 3x^2 + 5;\ C_4 : \alpha\frac{\partial T}{\partial x} = 0.$





$$\int_A N_1^i N_2^j N_3^k \, dA = \frac{i!\,j!\,k!}{(i+j+k+2)!}2A$$

$\alpha_1 = x_2 y_3 - x_3 y_2,\quad \beta_1 = y_2 - y_3,\quad \gamma_1 = x_3 - x_2,$

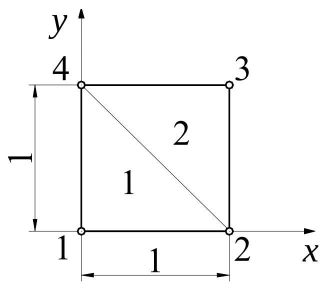
$\alpha_2 = x_3 y_1 - x_1 y_3,\quad \beta_2 = y_3 - y_1,\quad \gamma_2 = x_1 - x_3,$

$\alpha_3 = x_1 y_2 - x_2 y_1\quad \beta_3 = y_1 - y_2\quad \gamma_3 = x_2 - x_1$

$$N_i = \frac{1}{2A}\left(\alpha_i + x\beta_i + y\gamma_i\right),\ i = 1,3.$$

$$-\frac{\partial^2 T}{\partial x_i \partial x_i} = \frac{\rho q}{\alpha}.$$

Proračunski model:



$\mathbf{T}^T = \begin{bmatrix}T_1 & T_2 & T_3 & T_4\end{bmatrix}.$

Globalne čvorne varijable		1	2	3	4
Elementi, lokalne čv. var.	1	1	2	-	3
	2	-	1	2	3

$$\mathbf{K}^1 = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{K}^2 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

$$\mathbf{F}_S^1 = \int_A \mathbf{N}^T f \,\mathrm{d}A = \int_A \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\rho q}{\lambda}\,\mathrm{d}A.\quad \int_A N_1^i\, N_2^j\, N_3^k\, dA = \frac{i!\,j!\,k!}{\left(i+j+k+2\right)!}2A.$$

$$\int_A N_1\;\; dA = \frac{1!}{\left(1+2\right)!}2\cdot 0.5 = \frac{1}{6}.$$

$$\int_A N_2\;\; dA = \frac{1!}{\left(1+2\right)!}2\cdot 0.5 = \frac{1}{6}.$$

$$\int_A N_3\;\; dA = \frac{1!}{\left(1+2\right)!}2\cdot 0.5 = \frac{1}{6}.$$

$$\mathbf{F}_S^1 = \frac{1}{16}\frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.\quad \mathbf{F}_S^2 = \frac{1}{16}\frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\mathbf{R}_S^1 = \frac{1}{16}\frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.\quad \mathbf{R}_S^2 = \frac{1}{16}\frac{1}{6}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\mathbf{R}_S = \mathbf{R}_S^1 + \mathbf{R}_S^2 = \frac{1}{16}\frac{1}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

$$\mathbf{K} = \mathbf{K}^1 + \mathbf{K}^2 = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

Prirodni rubni uvjeti

$$C_2 : \mathbf{F}_b^1 = \int_l \mathbf{N}^T b_c \mathrm{d}l.$$

$$\mathbf{F}_b^2 = \int_0^1 \begin{bmatrix} 1-y \\ y \\ 0 \end{bmatrix} 6\;\mathrm{d}y = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}.$$

$$\mathbf{R}_b^2 = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}.$$

$$\mathbf{R} = \mathbf{R}_S + \mathbf{R}_b^2 = \frac{1}{96}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 1 \\ 290 \\ 289 \\ 2 \end{bmatrix}.$$

Osnovni rubni uvjeti

$$T_3 = 8, T_4 = 5.$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 2 \\ 290 \\ 289 \\ 1 \end{bmatrix},$$

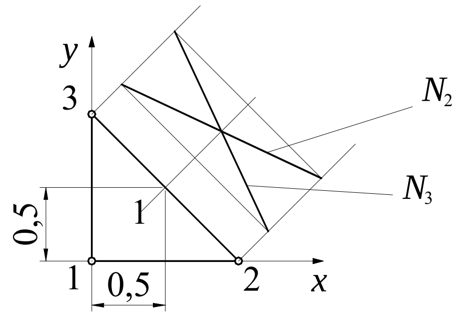
$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 2 \\ 290 \\ 289 \\ 1 \end{bmatrix} - 8\frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix},$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 2 \\ 290 \\ 289 \\ 1 \end{bmatrix} - 8\frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} - 5\frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 2+240 \\ 290+384 \\ 768 \\ 480 \end{bmatrix}.$$

$$\frac{1}{2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \frac{1}{96}\begin{bmatrix} 242 \\ 674 \end{bmatrix}.$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 8,0347 \\ 11,0278 \end{bmatrix}.$$

Točka u sredini ploče ima koordinate $(x, y) = (0,5, 0,5)$, tj. nalazi se na stranici koja spaja čvorove 2 i 4 proračunskog modela. Uzeto Budući da je za KE1 na toj stranici funkcija oblika N_1 jednaka nuli, a za KE2 na toj stranici funkcija oblika N_2 i na osnovi tablice podudaranja lokalnih i globalnih čvornih varijabli, može se uzeti za KE1 raspodjela funkcije nezavisne varijable pa stranici koja spaja globalne čvorove 2 i 4. Slijedi



$$\phi(0,5, 0,5) = N_2(0,5, 0,5)\phi_2^1 + N_3(0,5, 0,5)\phi_3^1.$$

$$\phi(0,5, 0,5) = \frac{1}{2}11,0287 + \frac{1}{2}5 = 8,014.$$