WHAT ARE MAGNITUDES AND HOW TO CALCULATE WITH THEM

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Math teachers detest "concrete numbers". They left them to physicists. Their problem is that standard mathematical education does not explain what are "concrete numbers" and how to calculate with them, although it is all obvious.(Obvious but mathematically tricky, that is something for physicists.) I will define "concrete numbers" as magnitudes and show that standard physicists' calculations with them are respectable mathematical calculations.

A magnitude is defined by an equivalence relation $\approx$ which is defined on objects (of interest) that are comparable ($<$) and additive ($+$). Furthermore, $+$ must be in harmony with $<$, and both $<$ and $+$ must be in harmony with $\approx$. It means that $\approx$ is a congruence.

Comparability means that $<$ is a linear order, and additivity means that $+$ is an associative and commutative operation with restricted difference, i.e. $a < b \to (\exists! c)\ a + c = b$.

(Alternatively, objects could be only additive, with $(\forall a,b)(\exists! c)(a+c = b \vee b+c = a)$. Then we define $a < b := a+c = b$.)

Now, given such a system of objects $(0, \approx, <, +)$, a system of corresponding magnitudes is defined as $(M, <, +) : = (0/\approx, </\approx, +/\approx)$. It is easy to see that, in mathematical parlance, M is a linearly ordered semigroup with restricted difference.

For example, think about objects (on a part of the surface of the earth) and their weights. An equivalence relation of "equally heavy" is introduced, by means of balance, and likewise a relation of "lighter then". Imagine also that two-objects can be joined to form a third. Equivalence classes of "equally heavy" are called weights. If we transfer "lighter then" and "joining" from the objects to the weight classes (i.e. weights), it is easy to check that we have got a system of magnitudes as previously defined (i.e. linearly ordered semi group with restricted difference).

Addition in a system of magnitudes M gives rise to a multiplication with positive integers n:

$$1\cdot a = a \qquad (n+1)\cdot a = n\cdot a + a.$$

If our system M is such that

$$(\forall a, b)(\exists n)\ a < n\cdot b,$$

we call it Archimedean. It is easy to prove that such a system contains neither 0 nor any additive inverses (in mathematical parlance it contains only positive elements).

By Hoelder-Cartan theorem (which I never met as a student of math, nor did I find it in analysis textbooks, although I reckon it the most important theorem on real numbers) Archemedean systems of magnitudes with no minimal magnitudes, are isomorphicaly and densely embeddable in R^+ (if they have minima they are isomorphic to Z^+). If a system of magnitudes is continuous, i.e. does not have empty Dedekind cuts, then it is isomorphic to R^+.

Furthermore, every linearly ordered semi group with restricted difference can be canonically extended to a linearly ordered group. By Hoelder-Cartan theorem, if this group is without minima it is isomorphicaly and densely embeddable in R, and if it is also continuous it is isomorphic to R. (It is isomorphic to Z if there is no minima.)

The embedding is constructed in the following way. We choose a unit u∈M. If na ≤ mu, for an a∈M, we say that a ≤ $\frac{n}{m}$ (similarly for a < $\frac{n}{m}$, a ≥ $\frac{n}{m}$ and a > $\frac{n}{m}$). In this way every a∈M determines a cut in Q, i.e. a real number r. Then we prove that $\mu : a \to r$ is the unique isomorphic embedding which sends u → 1.

This unique embedding, $\mu : M \to R$, is the measure generated by u. By definition, r(= μ(a)) measures a with respect to the corresponding unit u. We say that a = ur and in this way every a∈M is a real multiple of u.[1)]

In mathematical parlance, M is 1-dimensional real linear space.

From $\mu(ur) = r$ it follows that $ur = \mu^{-1}r$. So we can think of the unit u as the inverse of the corresponding measure μ, and then we usually call it a gauge.

In mathematical parlance, a unit u∈M is a base of the linear space M. Corresponding measure μ is the corresponding coordinatization of M.

For example, a possible unit of weight is kg (a specimen of which is held in Paris). The corresponding measure μ is such that

$$\mu(\text{kg } r) = r.$$

The inverse of this measure is the gauge μ^{-1} such that

$$\mu^{-1}r = \text{kg } r,$$

and in this sense kg = μ^{-1}. Gramm g is another unit/gauge, which we identify with the corresponding inverse measure μ_1^{-1}, such that

$$\text{kg } r = 1000 \text{ g}r \quad \text{i.e.} \quad \mu^{-1}(r) = 1000\, \mu_1^{-1}(r).$$

It means, as it should, that one gauge is 1000-fold another.

Systems of magnitudes M_1 and M_2 can be homomorphicaly related:

$$(1)\ a < b \rightarrow \mathbf{h}(a) < \mathbf{h}(b)$$

$$(2)\ \mathbf{h}(a+b) = \mathbf{h}(a) + \mathbf{h}(b).$$

(From (1) it follows that homomorphisms are really isomorphisms.) If we choose units/gauges μ_1, and μ_2, for M_1 and M_2, then $\mu_2\mathbf{h}\,\mu_1^{-1}$ is an isomorphism from R to R i.e. a linear function ℓ. From $\mu_2\mathbf{h}\,\mu_1^{-1} = \ell$ it follows that, for some constant c,

$$\mu_2(\mathbf{h}(a)) = c \cdot \mu_1(a),$$

which states that measure of $\mathbf{h}(a)$ is the constant multiple of the measure of a. This is the mathematical background of the rule of three.

For example, prices of some goods are isomorphisms between their weights w and their monetary values v, i.e. $\mathbf{pr}(w) = v$. If we measure weights in kg and monetary values in \$ (i.e. $\mu_1^{-1} = kg$ and $\mu_2^{-1} = \$$) then:

$$\$^{-1}(\mathbf{pr}(w)) = c \cdot kg^{-1}(w).$$

Of course, we commonly introduce real variables x = "a weight of goods in kg" and y = "the corresponding price in \$". Then we have a simpler equation

$$y = c \cdot x$$

which deals with "pure numbers" $x, y \in R$, and not with "concrete numbers" $w \in$ W(eights), $\mathbf{pr}(w) \in$ V(alues).

Mathematicians do not like to deal with "concrete numbers". They left them to physicists. The mathematicians' problem is that standard mathematical education does not explain what are "concrete numbers" or what is e.g. "corresponding price in \$", although it's all obvious. Obvious but mathematically tricky, that is something for physicists.

But we have mathematical explanations, we only do not teach them:

"Concrete numbers" are magnitudes e.g. W(eights).

"A weight of goods" is an $w \in W$.

"A weight of goods in kg" is $kg^{-1}(w) \in R$ (which means that $x = kg^{-1}(w)$).

"The corresponding price" is $\mathbf{pr}(w) \in$ V(alues).

"Corresponding price in \$" is $\$^{-1}(\mathbf{pr}(w)) \in R$ (which means that $y = \$^{-1}(\mathbf{pr}(w))$).

Similarly, the inverse rule of three can be founded on anti-isomorphisms:

$$(1)\ a < b \rightarrow \mathbf{h}(a) > \mathbf{h}(b)$$

$$(2)\ \mathbf{h}(na) = \frac{1}{n}\mathbf{h}(a)$$

If we choose units/gauges μ_1 and μ_2, for M_1 and M_2, then $\mu_2 \mathbf{h} \mu_1^{-1} = \mathbf{f}$ must be an anti-isomorphism from R to R, i.e. $\mathbf{f}(r) = c/r$, for some constant c. It follows that

$$\mu_2(\mathbf{h}(a)) = \frac{c}{\mu_1(a)}$$

which states that measure of **h**(a) is the constant multiple of the inverse of the measure of a. Of course, this is the mathematical background of the inverse rule of three.

Not only are magnitudes isomorphicaly related, they are also multiplied and devided by each other. By these operations we obtain new magnitudes, e.g. lenght^2 (length squared), weight/volume (weight per volume) and path/time (path per time); with the possible gauges, cm^2, kg/dm^3, cm/sec. We know these magnitudes as areas, specific weights and velocities respectively. Math teachers detest these compound "concrete numbers" even more then the simple ones. Divisions like

$$(400 \text{ m} / 50 \text{ sec}) / 10 \text{ sec} = 8 \text{ m/sec}^2$$

are shocking to most of them. But such calculations are absolutely rigorous and resistance against them is a pure ignorance. Let me explain.

From two systems of magnitudes A and B, a product can be constructed according to the rules which are those of tensor products of linear spaces: A·B consists of products a·b with $a \in A$ and $b \in B$, quotiented by

$$a\cdot(b_1+b_2) = a\cdot b_1 + a\cdot b_2$$

$$(a_1+ a_2)\cdot b = a_1\cdot b + a_2\cdot b$$

$$(ra)\cdot b = a\cdot(rb) \qquad r \in \mathrm{R}.$$

(Systems of magnitudes are 1-dimensional. It simplifies the matters a lot. For higher dimensional spaces a member of A·B is not necesseraly of the form a·b, it is a linear combination of such forms.)

It follows that with units $u \in A$ and $v \in B$ (and with corresponding measures μ and υ) every magnitude from A·B is of the form $(ur)\cdot(vs) = (u\cdot v)rs$. It means that rs is the measure of $(ru)\cdot(sv)$ with respect to the unit/measure $u\cdot v/\mu\cdot\upsilon$, i.e.

$$(\mu\cdot\upsilon)(ur \cdot vs) = r\cdot s = \mu(ur)\cdot\upsilon(vs)$$

For example

work = force · path and area = length · length,

and it is meaningful to write kg $r\cdot$ m s = kg·m rs and cm $r\cdot$cm s = $\text{cm}^2\, rs$.

Here, kg·m is the function, constructed from kg and m, that maps real numbers on works. Similarly, cm^2 maps real numbers on areas.

To explain the division of magnitudes we start from their reciprocals. For example, reciprocal lengths are known as dioptrics (with a unit m^{-1}) and reciprocal times are known as frequencies (with a unit sec^{-1} = Hertz). If we multiply a magnitude with its reciprocal magnitude we get a pure number. For example,

frequency · time = number of oscillations.

This product is linear in both magnitudes and according to the mathematical parlance of linear spaces (or modules) such spaces are duals of each other.

To every a∈M there correspond a dual a*∈M* such that a·a* = 1. If we choose dual units u∈M and u*∈M*, then a = u*r* and a* = u**s* and

$$a \cdot a^* = u \cdot u^*\, rs = 1 \quad \text{i.e.} \quad s = 1/r.$$

It means that measures of dual magnitudes (in dual units/gauges) are reciprocals of each others. This is the reason why dual magnitudes looks like reciprocals.

Most often dual/reciprocal magnitudes appears as factors in products. For example,

specific weight = weight/volume

velocity = path/time.

We compute with them in accordance with above rule. For example

$$\text{kg}\, r \cdot (\text{dm}^3 s)^{-1} = \text{kg}\cdot\text{dm}^{-3}\, rs^{-1} = \text{kg/dm}^3\, rs^{-1}$$

$$\text{m}\, r \cdot (\text{sec}\, s)^{-1} = \text{m}\cdot\text{sec}\, rs^{-1} = \text{m/sec}\, rs^{-1}$$

(Be aware that here sec^{-1} is the dual/reciprocal map of sec, which maps numbers on dual/reciprocal times; not an inverse of sec that maps numbers on times.)

Hence, this is the mathematical background of the physical calculations with magnitudes. It proves that it is mathematically rigorous and that mathematicians, and especially math teachers, should not detest it.

[1)] Of course, if $(\forall n)(\exists! b)nb = a$ then it is possible to define division of magnitudes by

$$\frac{a}{n} :=(\tau b)(nb = a).$$

Then, for every u∈M the hole Qu is defined and M is Archemedean if there is no magnitude that is smaller then and none that is larger them all of Qu. In that case every a∈M devides Qu in two parts

$$\{q\in Q\colon q\text{u} \le \text{a}\} \text{ and } \{q\in Q\colon \text{qu} > \text{a}\}.$$

This is a cut in Q, which defines $r\in$ R. We put a = ru and in this way every a$\in$M is a real multiple of u.