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# CANTOR'S THEOREM AND PARADOXICAL CLASSES

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In 1903 Russell announced the paradoxality of class $R = \{x\colon \sim x \in x\}$ and after his invention a lot of other paradoxical classes were invented. It is quite well known that Russell invented his paradoxical class by applying Cantor's theorem to the identity mapping of the universe of all classes onto itself, but this reasoning (as far as the author knows) has never been repeated in creation of (or at least in connection with) other paradoxes. For example there is no mention of it in Quine's presentations[2]) of his paradoxical classes

$$Q_1 = \{x\colon \sim(\exists y_1)\ (x \in y_1 \in x)\},\ Q_2 = \{x\colon\sim(\exists y_1)\ (\exists y_2)\ (x \in y_1 \in y_2 \in x)\} \text{ etc.},$$

although there is a very transparent similarity between Russell's $R$ and Quine's $Q_1, Q_2$, etc., which might suggest that the reasoning from Cantor's theorem may as well stand behind Quine's classes. It seems quite certain that this similarity guided Quine in his invention, but it also seems that he never connected (at least not publicly)[3]) the reasoning from Cantor's theorem with his paradoxical classes (although Cantor's theorem was one of the most fascinating themes of his NF, cf. [2]).

If we analyse this reasoning, which led from Cantor's theorem to Russell's paradoxical class, we realize at once that this same reasoning leads us to Quine's paradoxical classes, and also to several others not yet discovered. In one word we realize that the application of Cantor's theorem to the universe of all classes is a kind of uniform principle for generating paradoxical classes. The purpose of this paper is to explain this.

Let us first analyse Cantor's theorem. It asserts that every mapping (one-many relation) $\alpha$ between classes, and every class $a$, which is contained in the converse domain of $\alpha$, satisfy

(0) $\qquad \mathscr{P}(a) \subsetneqq \alpha\text{''}a.$

We will prove this assertion if (and only if) we succeed in defining class $C$[4]), within $\mathscr{P}(a)$, so that it satisfies

(1) $\qquad (x)\,(x \in a \supset \alpha\text{'}x \neq C).$

But if we are aware that $u \neq v$ means $(\exists z)\,(z \in v \equiv z \notin u)$, then we realize that we have to define class $C$ (within $\mathscr{P}(a)$) so that it satisfies

(2) $\qquad (x)\,(x \in a \supset (\exists y)\,(y \in C \equiv y \notin \alpha\text{'}x)).$

[1]) I would like to thank Prof. Quine, who kindly read an earlier draft and whose valuable remarks have been incorporated in the present article.

[2]) Cf. [3], pp. 128–130, and [4], p. 36.

[3]) After reading this paper Prof. Quine confirmed that he knew how Russell came to his paradox from Cantor's theorem (as he, after all, remarked in [4], p. 202) but that this was not how he came to $Q_1, Q_2$, etc.

[4]) Strictly speaking $C(\alpha, a)$, because $C$ depends on $\alpha$ and $a$ (we write $C$ for brevity).

Then it will certainly be enough to define class $C$ (within $\mathscr{P}(a)$) so that it satisfies

(3) $\qquad (x)\,(x \in a\ .\supset.\ x \in C \equiv x \notin \alpha' x)$.

And by this we have already defined such a class. Namely, class $C$ defined by

$$C = \{x\colon x \in a \;\&\; x \notin \alpha' x\}$$

satisfies our condition (1).[1])

Now we will explain how RUSSELL invented his paradoxical class $R$ by means of this, or some other, in its essence the same, reasoning. He took $\alpha$ to be the identity mapping $id$, which maps every class onto itself:

(4) $\qquad (x)\,(id' x = x)$.

This mapping is defined for all classes. This means, it is defined on universal class $V$ of all classes. RUSSELL must have realized that this strange class, which is such that

(5) $\qquad (x)\,(x \in V)$

must (by force of (5)) satisfy

(6) $\qquad \mathscr{P}(V) \subseteqq V$.

On the other hand, we have (by (5))

(7) $\qquad id'' V = V$.

Combining (6) and (7) we get

(8) $\qquad \mathscr{P}(V) \subseteqq id'' V$

contrary to CANTOR's theorem.

But, given any mapping $\alpha$ and any class $a$ contained in converse domain of $\alpha$, we have succesfully defined class $C(\alpha, a)$, which led us towards the proof of CANTOR's theorem. Confronted with the conflict between (0) and (8) it is natural to explore what happens with (for CANTOR's theorem) crucial class $C(\alpha, a)$ when $\alpha = id$ and $a = V$.

By (3) and (4) $C(id, V)$ satisfies

(9) $\qquad (x)\,(x \in V\ .\supset.\ x \in C(id, V) \equiv x \notin x)$,

which is, because of (5), equivalent to

(10) $\qquad (x)\,(x \in C(id, V) \equiv x \notin x)$.

So, the paradoxical situation, with which we were confronted when choosing $id$ and $V$ as $\alpha$ and $a$, respectively, led us exactly to RUSSELL's paradoxical class:

(11) $\qquad R = C(id, V)$.

But considering (0), (6) and (7) we immediately realize that we shall be confronted with the same paradoxical situation for any mapping $\alpha$, defined on $V$, which satisfies

(12) $\qquad \alpha'' V = V$,

and that, in this case, $C(\alpha, V)$ will be a paradoxical class as well. Namely, class $C(\alpha, V)$ (which we will call $C$, for a while) satisfies

(13) $\qquad (x)\,(x \in C \equiv x \notin \alpha' x)$

[1]) Our analysis of what is to be proved thus ends by the proof itself.

because of (5). It follows by (12) that

(14) $(\exists y)\, \alpha\text{'}y = C$

and we will call one of the classes whose existence is asserted by (14) $C'$. Then we have

(15) $\alpha\text{'}C' = C$.

It follows then from (13) and (15) that

(16) $C' \in C \equiv C' \notin C$

and here we have shown the above asserted paradoxality of $C(\alpha, V)$.

The crucial thing was that $\alpha$, which is defined on $V$, satisfies (12) and it is easy to find a lot of mappings defined on $V$ which satisfy (12):

Our first example is the union $\bigcup$ and its iterations $\bigcup^2, \bigcup^3$ etc. They are indeed defined for all classes, and every class in itself is a union of some other class.

(17) $a = \bigcup\{a\}$[1])

for example. Similarly we have

(18) $a = \bigcup^2\{\{a\}\}, \qquad a = \bigcup^3\{\{\{a\}\}\},$ etc.

Now, it is easy to show that

(19) $C(\bigcup, V) = Q_1, \qquad C(\bigcup^2, V) = Q_2, \qquad C(\bigcup^3, V) = Q_3,$ etc.[2])

Indeed, $C(\bigcup, V)$ satisfies (by (13))

(20) $(x)\,(x \in C(\bigcup, V) \equiv x \notin \bigcup x)$

which, written completely, reads

(21) $(x)\,(x \in C(\bigcup, V) \equiv \sim(\exists y)\,(x \in y \in x))$.

Similarly $C(\bigcup^2, V)$ satisfies

(22) $(x)\,(x \in C(\bigcup^2, V) \equiv x \notin \bigcup^2 x)$,

which, written completely, reads

(23) $(x)\,(x \in C(\bigcup^2, V) \equiv \sim(\exists y_1)\,(\exists y_2)\,(x \in y_1 \in y_2 \in x))$,

and so on.

Finally it will be interesting to generate some new paradoxical classes by choosing appropriate mappings $\alpha$. Natural candidates will be intersection $\bigcap$ and its iterations $\bigcap^2, \bigcap^3$, etc., which are defined for all classes and which satisfy (12) because of

(24) $a = \bigcap\{a\}, \qquad a = \bigcap^2\{\{a\}\}, \qquad a = \bigcap^3\{\{\{a\}\}\},$ etc.

Then $C(\bigcap, V)$ satisfies (by (13))

(25) $(x)\,(x \in C(\bigcap, V) \equiv x \notin \bigcap x)$,

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[1]) We use the standard term $\bigcup x$ instead of $\bigcup\text{'}x$, and we do not use the term $\bigcup\text{''}x$ at all. The same convention is used for $\bigcup^2, \bigcup^3$, etc., $\bigcap, \bigcap^2, \bigcap^3$, etc. and $^-$.

[2]) We may say: the relation that RUSSELL's paradoxical class bears to identity is the same as the one that QUINE's paradoxical classes bear to union and its iterations.

which, written completely, reads

(26) $(x)\,(x \in C(\bigcap, V) \equiv \sim(y)\,(y \in x \supset x \in y))$.

Similarly $C(\bigcap^2, V)$ satisfies

(27) $(x)\,(x \in C(\bigcap^2, V) \equiv \sim x \in \bigcap^2 x)$,

which, written completely, reads

(28) $(x)\,(x \in C(\bigcap^2, V) \equiv \sim(y_1)\,((y_2)\,(y_2 \in x \supset y_1 \in y_2) \supset x \in y_1)$

or equivalently

(29) $(x)\,(x \in C(\bigcap^2, V) \equiv \sim(y_1)\,(\exists y_2)\,((y_2 \in x \supset y_1 \in y_2) \supset x \in y_1)$, etc.

Thus the new paradoxical classes are:

$S_1 = \{x\colon \sim(y)\,(y \in x \supset x \in y)\}$,

$S_2 = \{x\colon \sim(y_1)\,(\exists y_2)\,((y_2 \in x \supset y_1 \in y_2) \supset x \in y_1)\}$,

$S_3 = \{x\colon \sim(y_1)\,(\exists y_2)\,(y_3)\,(((y_3 \in x \supset y_2 \in y_3) \supset y_1 \in y_2) \supset x \in y_1)\}$,

$S_4 = \{x\colon \sim(y_1)\,(\exists y_2)\,(y_3)\,(\exists y_4)\,((((y_4 \in x \supset y_3 \in y_4) \supset y_2 \in y_3) \supset y_1 \in y_2) \supset x \in y_1)\}$,

etc.

If we suppose that every class has a complement, we have one more example of a paradoxical class by taking complementation $^-$ as mapping $\alpha$. In this case, where $^-$ is defined for all classes, it satisfies (12) as well, because of

(30) $a = \bar{\bar{a}}$.

Then $C(^-, V)$ satisfies (by (13))

(31) $(x)\,(x \in C(^-, V) \equiv x \notin \bar{x})$,

which, written completely, reads

(32) $(x)\,(x \in C(^-, V) \equiv x \in x)$.

The new paradoxical class is the complement of RUSSELL's $R$

(33) $C(^-, V) = \bar{R} = \{x\colon x \in x\}$.

Problems and conjectures.

1. In our presentation of paradoxical classes the paradox is always derived by using the appropriate construct $C'$; and it is usually a set theoretical construct. For example, the paradoxality of $Q_1$ is proved by deriving the contradiction

$$\{Q_1\} \in Q_1 \equiv \{Q_1\} \notin Q_1$$

(cf. (15), (16) and (17)) from the assertion

$$(x)\,(x \in Q_1 \equiv x \notin \bigcup x),$$

which defines $Q_1$ (cf. (13)). Similarly from $Q_2$, $Q_3$, etc. we got

$$\{\{Q_2\}\} \in Q_2 \equiv \{\{Q_2\}\} \notin Q_2, \qquad \{\{\{Q_3\}\}\} \in Q_3 \equiv \{\{\{Q_3\}\}\} \notin Q_3, \quad \text{etc.}$$

In the same way, from the assertions which define $S_1$, $S_2$, etc. we get the contradictions

$$\{S_1\} \in S_1 \equiv \{S_1\} \notin S_1, \qquad \{\{S_2\}\} \in S_2 \equiv \{\{S_2\}\} \notin S_2, \quad \text{etc.}$$

But in the case of QUINE's classes it is possible to derive a contradiction strictly logically. Namely, we can prove[1]), within first order logic (without using any set theoretical construct like $\{\ \}$) that

$$\sim(\exists y)\,(x)\,(x \in y \equiv\ \sim(\exists u)\,(x \in u \in x)),$$
$$\sim(\exists y)\,(x)\,(x \in y \equiv\ \sim(\exists u)\,(\exists v)\,(x \in u \in v \in x)), \quad \text{etc.}$$

This means that we can prove the above sentences without using any set theoretical properties of $\in$ (i.e. we could prove the above sentences even when replacing $\in$ by any two-place predicate parameter $P$). The problem is: is it possible to prove the paradoxality of $S_1, S_2$, etc. strictly logically?

In an earlier version of this paper I conjectured the positive answer because "it would appear quite strange that a strictly logical proof, possible for the *union*-generated $Q_1, Q_2$, etc., is not possible for the *intersection*-generated $S_1, S_2$, etc.". But (to my surprise) QUINE proved the conjecture false. His argument is as follows:

The question is whether

$$(\exists z)\,(x)\,(x \in z \equiv\ \sim(y)\,(y \in x \supset x \in y))$$

is false purely by quantification theory. In other words, disinterpreting epsilon, the question is whether

$$(\exists z)\,(x)\,(Fxz \equiv\ \sim(y)\,(Fyx \supset Fxy))$$

is unsatisfiable. It is satisfiable in natural numbers by interpreting "$F$" as "$\geqq$" and taking $z$ as $0$.

QUINE also remarked that this asymmetry between $S_1, S_2$, etc. and $Q_1, Q_2$, etc. is not surprising when we reflect that the paradox of $\{x\colon x \in x\}$, unlike that of $\{x\colon x \notin x\}$, is likewise not strictly logical. The schema "$(\exists z)\,(x)\,(x \in z \equiv x \in x)$" is satisfiable, again, by the above interpretation.

2. The second problem is to define a paradoxical class (or to point to some already defined one), the paradoxality of which cannot be generated by our uniform principle. MIRIMANOFF's class[2]) $M = \{x\colon x \text{ is grounded}\}$ is worth studying in that respect. The conjecture is that there is no mapping $\alpha$ which generates $M$.[3])

3. Our discussion was informal, so as to be understandable to a wider audience, but it would be easy to formalize it within a (formal) set theory which admits the universal class $V$. Such is QUINE's NF, for example. If we formalize our reasoning within NF, it follows that every condition, which generates a paradoxical class, because it is equivalent to $x \notin \alpha' x$ for some appropriate mapping $\alpha$ definable in NF, and which we shall call *$\alpha$-condition* for brevity, must be unstratified, if NF is consistent. In other words, every $\alpha$-condition is an unstratified condition, if NF is consistent.

The problem: Is every unstratified condition an $\alpha$-condition? Conjecture: It is not.

[1]) Cf. [3], pp. 128–130, and [4], p. 36.

[2]) Cf. [1] and [5].

[3]) This would be the first step towards a classification of set-theoretical paradoxes. On one side there would be the paradoxes generated by our uniform principle (and within them a further division between "strictly logical" and "purely set-theoretical" (cf. 1.) would be possible) and on the other side there would be the paradoxes not generated by that principle.

If this conjecture is right, there are two possibilities. First, there are conditions that are both unstratified and paradoxical and which are not $\alpha$-conditions.[1]) (These conditions, when used as clauses in the principle of abstraction, yield a contradiction.) Second, there are no conditions that are both unstratified and paradoxical, and which are not $\alpha$-conditions. All such unstratified conditions could be used as clauses in the principle of abstraction without yielding any contradiction. If the second possibility comes true[2]), then here opens a way of further relaxations as to the admissibility of conditions which can be used as clauses in the principle of abstraction.[3])

## References

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[1]) This possibility is suggested by the conjecture in 2.
[2]) But this is not suggested by the conjecture in 2.
[3]) The step from the theory of types to NF could thus be followed by the another one. But Quine justifiably warned that the liberalization would not be immediate, because the criterion "not equivalent to $x \notin \alpha' x$ for any definable $\alpha$" is not effectively testable and hence not directly available to rules of proof.