Passive and active vibration isolation systems using inerter

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Abstract

This paper presents a theoretical study on passive and active vibration isolation schemes using inerter elements in a two degree of freedom (DOF) mechanical system. The aim of the work is to discuss basic capabilities and limitations of the vibration control systems at hand using simple and physically transparent models. Broad frequency band dynamic excitation of the source DOF is assumed. The purpose of the isolator system is to prevent vibration transmission to the receiving DOF. The frequency averaged kinetic energy of the receiving mass is used as the metric for vibration isolation quality. It is shown that the use of inerter element in the passive vibration isolation scheme can enhance the isolation effect. In the active case, a feedback disturbance rejection scheme is considered. Here, the error signal is the receiving body absolute velocity which is directly fed to a reactive force actuator between the source and the receiving bodies. In such a scheme, the so-called subcritical vibration isolation problems exist. These problems are characterised by the uncoupled natural frequency of the receiving body larger than the uncoupled natural frequency of the source body. In subcritical vibration isolation problems, the performance of the active control is limited by poor stability margins. This is because the stable feedback gain is restricted in a narrow range between a minimum and a maximum. However, with the inclusion of an inerter in the isolator, one of the two stability margins can be opened. This enables large, theoretically unlimited negative feedback gains and large active damping of the receiving body vibration. A simple expression for the required inertance is derived.

Keywords: Vibration Isolation, Inerter, Active Vibration Control, Direct Velocity Feedback, Stability of Active Control Systems, Optimization of Vibration Control Systems

1. Introduction

Inerter is a one port element in mechanical networks which resists relative acceleration across its two terminals [1], [2]. The coefficient of this resistance is called inertance and is measured in kilograms. An
appealing property of inerters is that they can be designed and realised in practice having their inertance significantly larger than their mass [1], [2]. This opens many interesting possibilities so that many authors reported on how to design and use inerters to suppress mechanical vibrations [1]–[23].

The concept of “relative mass” has been considered in the past in connection with mechanical–electrical analogies by Schönfeld [24]. He mentioned the possibility of a two-terminal mechanical inertance and gave a rudimentary scheme of a physical realisation of the concept. Smith [1], and Smith and Wang [2] developed this idea by investigating how to design such a device in practice and pointed out a number of peculiarities that the new element brings into a mechanical network. The authors described the characteristic phase lead property which cannot be achieved with conventional passive struts consisting of springs and dampers only, and instilled that inerter is the analogue of the capacitor element in electrical networks [2]. Therefore, adding the inerter to classical dampers and springs fills an empty niche enabling a complete synthesis of passive mechanical networks [2]–[4], [24].

Smith and Wang designed their inerter using a plunger sliding in a cylinder which drives a flywheel through a rack, pinion and gears [2]. In this design the inertance can be set by the choosing the gear ratio. Such a realisation should be viewed as approximating its mathematical ideal in a similar way that real springs, dampers, capacitors, etc. approximate their mathematical ideals [2]. In other words, effects such as friction, stick-slip of the gear pairs, or the elasticity of the gears and connecting rods are inevitably present in gear-train based inerter constructions. Other physical realisations of inerters have been proposed as well. For example, an electromagnetic transducer (voice coil, linear motor) can be shunted with an electrical impedance consisting of a capacitance connected in series to a parallel resistance-inductance pair. If the total shunt impedance is properly tuned, then the whole electromechanical network theoretically behaves exactly as if it incorporated an ideal inerter mounted in series with a parallel spring damper-pair [6]. A problem in this realisation is that voice coils are characterised by an inherent electric resistance of the wire in the coil. This resistance causes the dimensionless electromechanical coupling coefficient of the transducer to downscale rather unfavourably [25]–[27]. As a result, unrealistically large scale electromagnetic transducers would be needed to synthesise a usable inerter by means of entirely passive electrical shunt circuits. This can be overcome by actively compensating for the coil resistance [6]. A number of “negative impedance” electrical circuit designs comprising operational amplifiers, which could be used for this purpose can be found in [28]. However, such an approach is active which on one hand requires energy and on the other a careful regard of the stability and robustness of the system. For these reasons, self-powered configurations employing a simultaneous active control and energy harvesting have been considered to synthesise mechatronic inerters [6]. Another type of mechatronic inerter utilises a rotary DC motor shunted with an appropriate electrical circuit [7]. This is in order to supplement the mechanical inertance associated with the rotor moment of inertia with additional electrically synthesised inertance [7]. An inertance-like behaviour can also be accomplished through a scheme in which hydraulic fluid is accelerated [8], [9]. This can be achieved with a piston which pushes the fluid through a helical channel [9]. This design involves relatively
large parasitic damping so that the device is best modelled by considering a nonlinear damper in parallel to the idealised inerter [9].

Inerters can be very useful in vibration isolation systems. In this sense, many authors focused their efforts on improving vehicle suspension systems using inerters [2], [10]-[13]. Further applications of inerters include vibration isolation in civil engineering structures, such as multi-storey buildings under earthquake base excitation [14]. In vibration isolation problems it is often necessary to tune the impedance of the isolator elements based on some optimisation criteria. This can be done by either minimising maxima of the response (minimax or $H_\infty$ optimisation), or by minimising the energy in the response signals ($H_2$ optimisation) [15]. Inerters can also be very useful in vibration absorber systems. Performance of vibration absorbers, especially Tuned Mass Dampers (TMDs) is known to very much depend on the proof mass added to a primary structure to reduce its vibration. This mass is added to structures exclusively to control their vibrations, so it is penalised in lightweight automotive and aerospace applications [16], [17]. In this context the use of inerter elements can be interesting given the fact that their inertance can be significantly larger than their mass. Consequently a number of new concepts have arisen. These include tuned inerter damper (TID), tuned mass–damper–inerter (TMDI), and inerter–based dynamic vibration absorber (IDVA) [18]-[22]. In these systems the working frequency of the absorber can be tuned by changing the inertance. In particular, it can be reduced without increasing the physical mass of the vibration absorber while preserving the static stiffness of the absorber suspension spring. Various applications have been considered using tuned inerter dampers including vibration reduction of cables in cable-stayed bridges [18], [19].

Dynamic vibration absorbers can be made active. Active vibration absorbers can be realised using inertial actuators with a velocity or velocity + displacement feedback control scheme [29]-[36]. Normally, inertial actuators must be designed with a low mounted natural frequency [29]-[36]. This requires either large inertial mass or soft suspension stiffness. Both is hard to realise in practice since the mass must not be too large as this would add too much weight to the structure, and the stiffness cannot be too small due to large sags in case of constant accelerations (gravity, vehicle manoeuvring). The low natural frequency also limits the applicability of inertial actuators in cases of structures rotating at a high speed which exposes co-rotating actuators to large centrifugal forces [37]-[39].

Considering now the use of inerters in active vibration absorber systems, Zilletti investigated a system in which the inerter is attached in parallel with the suspension spring, damper, and the actuator [23]. The author has shown that in this way it is possible to reduce the blocked natural frequency of the actuator without adding to the actual proof mass, apart from the relatively small mass added by the inerter construction. This approach has been shown not only to increase the range of frequencies where the active control can be achieved, but also to improve the stability and the robustness of the active control scheme which uses the inertial actuator to develop the control force [23]. Zilletti considered only an idealised inerter element, which neglects the inertial, stiffness and damping effects of the gearing mechanism that converts axial relative motion at the terminals of the inerter into angular motion of the inerter wheels. However, Kras and Gardonio
considered the effective weight and dynamics effects of an inerter element composed by a single flywheel which is either pinned or hinged to the base mass or to the proof mass of the actuator [40].

In this paper an active vibration isolation problem is considered. It is shown that the use of inerter can significantly improve the stability and performance of the active vibration isolation system in certain situations. In particular, it is shown analytically on a simplified model problem that the use of inerter enables successful active vibration isolation in a family of mechanical systems that are otherwise difficult to control. This family of system has been referred to as subcritical 2 DOF systems. Subcritical systems are those characterised by the natural frequency of the receiving body larger than the natural frequency of the source body. In such vibration isolation problems the use of inerter is shown to stabilise the feedback loop and therefore to enable a remarkable active vibration isolation effect. In addition to the active vibration isolation system, several inerter-based and inerter-free passive isolator schemes are proposed and analysed, with the aim of establishing fair benchmarks for the evaluation of the performance of the active isolators studied later in the paper.

The paper is structured into six sections. In the second section, the physical and mathematical models are presented and the model problem is postulated. In section 3 a benchmark passive vibration isolation scheme not employing the inerter is discussed. In section 4 a benchmark passive vibration isolation scheme employing the inerter is analysed. Finally, in section 5 a comprehensive stability and performance analysis of the active vibration isolation scheme is given. This analysis indicates the subcritical family of vibration isolation systems that requires the use of inerters in the isolator to have stable and performant active vibration isolator. In order to ensure a fair comparison among all active and passive configurations, the performance of the vibration isolation is measured through a unified criterion which is the mean kinetic energy of the receiving body. In each system, either active of passive, tuneable parameters are adjusted in order to minimise the kinetic energy of the receiving body per unit, spectrally white, dynamic excitation of the source body.

2. Mathematical model

In this section the mathematical model of an inerter-based active vibration isolation system is formulated. As shown in Fig. 1., the problem studied is represented by a lumped parameter two degree of freedom (DOF) mechanical system. The system consists of two masses $m_1$ and $m_2$ coupled by a spring $k_2$, a viscous damper $c_2$ and an inerter of inertance $b_2$. The inerter produces a force proportional to the relative acceleration between masses $m_1$ and $m_2$. The two masses are attached to fixed reference bases via the two mounting springs $k_1$ and $k_3$. The lower mass $m_1$ is excited by the disturbance force $F_1$. It is assumed that the force $F_1$ has characteristics of an ideal white noise and that the power spectral density (PSD) of the force equals one over all frequencies.

In this study the purpose of the vibration isolation system is to reduce vibrations of mass $m_2$ which are due to the forcing $F_1$ acting on the mass $m_1$. Therefore, a structure approximated by the mass $m_1$ and spring $k_1$ is
referred to as the source body, and a structure characterised by the mass $m_2$ and stiffness $k_3$ is referred to as the receiving body (Fig. 1).

![Diagram of a two degree of freedom active vibration isolation system](image)

**Fig. 1. The two degree of freedom active vibration isolation system**

Such lumped parameter approximation may be representative of a system of more complicated nature, incorporating structures with distributed mass and stiffness parameters. For example, the modal mass and stiffness of the fundamental mode of a flexible rectangular source panel can be represented through the mass $m_1$ and stiffness $k_1$. Similarly, the mass $m_2$ and stiffness $k_3$ can represent the modal mass and stiffness corresponding to the fundamental mode of a flexible radiating panel. Finally, the stiffness $k_2$ between the two masses could represent a coupling impedance associated with the breathing mode of an air cavity between the two panels. In such a way the simplified 2 DOF model could be used to describe the low-frequency dynamic behaviour of acoustically coupled double panels as discussed in, for example [41]. Other systems may also be representable by the general configuration shown in Fig. 1. [29], [42]-[44]. In case a more detailed and accurate analysis is required, attention should be paid to the influence of higher order residual modes, see for example [45].

The active part of the vibration isolation system is realised through a skyhook damping unit [44], [46]. The skyhook damper consists of a reactive actuator, a velocity sensor, and a feedback loop between the output of the sensor and the input to the actuator. The actuator is mounted in parallel with the passive part of the isolation system (spring, dashpot and inerter) with its terminals also attached to the two masses (Fig. 1.). The velocity sensor is mounted onto mass $m_2$ in order to realise a disturbance rejection control scheme. In this scheme the actuator is driven with a signal proportional to the negative absolute velocity of the receiving body amplified by a constant control gain $g$. Idealised sensor-actuator transducers are assumed. Thus the feedback gain $g$ has physical dimension of Ns/m and could be referred to as the active damping coefficient. Practical velocity sensors are normally realized using standard accelerometers with time-integrated outputs. The cut-off frequency of the integration circuit is usually chosen low, so that in the frequency range between the cut-off frequency of the integrator and the blocked natural frequency of the accelerometer, the time-
integrated output of the accelerometer is proportional to velocity [25], [29], [31], [32]. Also, an advanced MEMS velocity sensor with internal velocity feedback has been proposed in [47].

The actuator force $F_A$ is given by

$$ F_A = -g \ddot{x}_2. $$

(1)

The equations of motion are

$$
\begin{align}
(m_1 + b_1) \dddot{x}_1 - b_2 \ddot{x}_2 + c_2 \dot{x}_1 - (c_2 + g) \ddot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 &= F_1, \\
-b_2 \dddot{x}_1 + (m_2 + b_2) \ddot{x}_2 - c_2 \dot{x}_1 + (c_2 + g) \ddot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 &= 0.
\end{align}
$$

(2a,b)

The equations of motion Eq. (2a,b) can be written in the matrix form as

$$
M \dddot{x} + C \dot{x} + K x = F,
$$

(3)

where $M$ is the mass matrix, $K$ is the stiffness matrix, $C$ is the damping matrix, $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ are the displacement, velocity and acceleration column vectors respectively, and $F(t)$ is excitation column vector. These matrices/vectors are given by the following expressions

$$
M = \begin{bmatrix}
m_1 + b_2 & -b_2 \\
-b_2 & m_2 + b_2
\end{bmatrix},
C = \begin{bmatrix}
c_1 + c_2 & -c_2 - g \\
-c_2 & c_2 + c_3 + g
\end{bmatrix},
K = \begin{bmatrix}
k_1 + k_2 & -k_2 \\
-k_2 & k_2 + k_3
\end{bmatrix},
$$

(4a-c)

$$
x = \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix},
F = \begin{bmatrix}
F_1(t) \\
0
\end{bmatrix},
$$

(5a-b)

where the parameters/functions in the matrices/vectors are as indicated in Fig. 1. Note that the gain $g$ generates diagonally asymmetric active damping terms in the system damping matrix $C$. Throughout this study, the damping of the source and receiving structures is assumed to be light. Thus the effects of dampers between the source mass $m_1$ and the ground and between the receiving mass $m_2$ and the ground are neglected i.e. $c_1 \approx c_3 \approx 0$. This enables significantly less complex mathematical derivations in the forthcoming parts of the study. Furthermore it leads to a more transparent model regarding the physics governing the system dynamical behaviour. Nevertheless, the main results are cross-checked with results using a full damping model where the influence of dampers $c_1$ and $c_3$ is not neglected. These results with the full damping model can be found in Appendix of the paper.

Assuming a simple harmonic excitation and expressing the excitation and the steady-state response in the exponential form $F(t) = \hat{F} e^{j\omega t}$ and $x(t) = \hat{x} e^{j\omega t}$, where $j = \sqrt{-1}$, Eq. (3) can be written as

$$
S(j\omega)x(j\omega) = F(j\omega).
$$

(6)

where $S(j\omega)$ is the dynamic stiffness matrix with the following form

$$
S(j\omega) = -\omega^2 M + j\omega C + K.
$$

(7)

Solution of Eq. (6) can be obtained by inversion of the dynamic stiffness matrix $S(j\omega)$ as

$$
x(j\omega) = S^{-1}(j\omega)F(j\omega).
$$

(8)
Differentiating Eq. (8) in order to obtain velocities results in expression

\[ \mathbf{x}(j\omega) = \mathbf{Y}(j\omega) \mathbf{F}(j\omega), \]  

(9)

where \( \mathbf{x}(j\omega) = j\omega \mathbf{x}(j\omega) \) is the velocity vector and \( \mathbf{Y}(j\omega) = j\omega \mathbf{S}^{-1}(j\omega) \) is the mobility matrix representing four frequency response functions (FRFs) between velocities and forces. By taking \( \mathbf{M}, \mathbf{K} \) and \( \mathbf{C} \) matrices from Eq. (4a-c), the steady-state complex response can be expressed in terms of the two driving points and two transfer mobilities as

\[
Y_{1z}(j\omega) = \frac{(j\omega)^3 (m_2 + b_i) + (j\omega)^2 (c_j - g) + (j\omega)(k_2 + k_i)}{(j\omega)^3 [(b_2 + m_2)m_2 + b_2m_2] + (j\omega)^2 [(c_j + g)m_2 + c_2m_2] + (j\omega) [(c_j + g)k_2 + c_2k_i] + (k_2 + k_i)k_2 + k_2k_i}, \tag{10a}
\]

\[
Y_{1y}(j\omega) = \frac{(j\omega)^4 b_1 + (j\omega)^3 (c_j - g) + (j\omega)k_2}{(j\omega)^4 [(b_2 + m_2)m_2 + b_2m_2] + (j\omega)^3 [(c_j + g)m_2 + c_2m_2] + (j\omega) [(c_j + g)k_2 + c_2k_i] + (k_2 + k_i)k_2 + k_2k_i}, \tag{10b}
\]

\[
Y_{2z}(j\omega) = \frac{(j\omega)^3 (m_2 + b_i) + (j\omega)^2 (c_j - g) + (j\omega)(k_2 + k_i)}{(j\omega)^3 [(b_2 + m_2)m_2 + b_2m_2] + (j\omega)^2 [(c_j + g)m_2 + c_2m_2] + (j\omega) [(c_j + g)k_2 + c_2k_i] + (k_2 + k_i)k_2 + k_2k_i}, \tag{10c}
\]

\[
Y_{2y}(j\omega) = \frac{(j\omega)^4 b_1 + (j\omega)^3 (c_j - g) + (j\omega)k_2}{(j\omega)^4 [(b_2 + m_2)m_2 + b_2m_2] + (j\omega)^3 [(c_j + g)m_2 + c_2m_2] + (j\omega) [(c_j + g)k_2 + c_2k_i] + (k_2 + k_i)k_2 + k_2k_i}, \tag{10d}
\]

where \( Y_{ij} = \dot{x}_i / F_j \) is a mobility function of the system, representing a velocity of the mass \( i \) due to a unit force at the mass \( j \). If \( i = j \) then the corresponding FRF is referred to as a driving point mobility, otherwise it is referred to as a transfer mobility.

The transfer mobility \( Y_{2z} \), representing the velocity response of the receiving body per unit forcing of the source body, is used to assess the quality of the vibration isolation throughout this paper. With the aim of more general approach, mobility \( Y_{2z} \) in Eq. (10c) can be expressed in the following dimensionless form

\[
Y_{2z}(j\Omega) = \frac{B_0 + (j\Omega)B_1 + (j\Omega)^2 B_2 + (j\Omega)^3 B_3}{A_0 + (j\Omega)A_1 + (j\Omega)^2 A_2 + (j\Omega)^3 A_3 + (j\Omega)^4 A_4}, \tag{11}
\]

where coefficients \( A_0...A_4 \) and \( B_0...B_3 \) are given by

\[
A_0 = \mu_1 (\alpha \beta \mu_1 + \alpha + \beta) \quad B_0 = 0
\]

\[
A_1 = 2\eta_1 (\beta \mu_1 + \lambda + 1) \quad B_1 = \alpha \mu_1
\]

\[
A_2 = \mu_1 (\alpha \mu_1 + \beta \mu_1 + \alpha + \beta + \mu_1 + 1) \quad B_2 = 2\eta_2 \quad (\text{12a-i})
\]

\[
A_3 = 2\eta_3 (\mu_1 + \lambda + 1) \quad B_3 = \mu_1 \mu_2
\]

\[
A_4 = \mu_1 (\mu_1 \mu_2 + \mu_2 + 1)
\]

where

\[
\alpha = \left( \frac{\Omega_2}{\Omega_1} \right)^2 \quad \beta = \left( \frac{\Omega_2}{\Omega_1} \right)^2 \quad \eta_2 = \frac{c_2}{2\sqrt{m_1 k_1}} \quad \lambda = \frac{g}{c_2} \quad \mu_1 = \frac{m_2}{m_1} \quad \mu_2 = \frac{b_2}{m_2} \quad \Omega = \frac{\omega}{\Omega_1} \quad (\text{13a-g})
\]
and $Y_{21} = m_1 \Omega Y_1$ is now the dimensionless transfer mobility. Throughout the rest of the paper, it is assumed that $m_1$ and $\Omega_1$ are constant values, used for scaling the transfer mobility function $Y_{21}$ to convenient dimensionless form.

In Eqs. (13a-g), $\alpha$ and $\beta$ are squared natural frequency ratios, $\eta_2$ is the damping ratio, $\lambda_2$ is the feedback gain normalised with respect to the passive damping coefficient, and $\mu_1$ and $\mu_2$ are the mass and inertance ratios respectively. Furthermore, $\Omega$ is dimensionless circular frequency normalised with respect to the natural frequency of the uncoupled source body $\Omega_1$ (as if the source body was uncoupled by removing spring $k_2$), $\Omega_3$ is the natural frequency of the uncoupled receiving body (as if the receiving body was uncoupled by removing spring $k_2$), and $\Omega_2$ is the natural frequency of the receiving body as if it was attached to a fixed reference base through the spring of stiffness $k_2$ only. The three natural frequencies $\Omega_1, \ldots, \Omega_3$ are thus

$$\Omega = \sqrt{\frac{k_1}{m_1}}, \quad \Omega_2 = \sqrt{\frac{k_2}{m_2}}, \quad \Omega_3 = \sqrt{\frac{k_3}{m_3}}. \quad (14a-c)$$

Given that the excitation force $F_1$ with unit PSD has been assumed, the specific kinetic energy of the receiving body (per unit mass, per unit excitation force) can be calculated as

$$I_k = \int_{-\infty}^{\infty} |Y_{21}(j\Omega)|^2 \, d\Omega,$$  

(15)

according to the Parseval's identity. The specific kinetic energy index $I_k$ is used throughout this study as a measure of the performance of broad frequency band vibration isolation. The objective is to minimise this quantity of all vibration isolation systems analysed in the paper.

The specific kinetic energy index in Eq. (15) can according to [48] be calculated as

$$I_k = \pi \frac{A_o B_1^2 \left( A_o A_3 - A_2 A_4 \right) + A_o A_3 \left( 2B_1 B_2 - B_2^2 \right) - A_o A_4 \left( B_1^2 - 2B_0 B_2 \right) + A_4 B_0^2 \left( A_o A_4 - A_2 A_3 \right)}{A_o A_3 \left( A_o A_3^2 + A_4^2 A_1 - A_4 A_3 A_1 \right)}.$$  

(16)

Substituting coefficients $A_0 \ldots A_4$ and $B_0 \ldots B_3$ from Eq. (12) into Eq. (16) yields

$$I_k = 2\pi \left[ \frac{1}{4 \mu_2 \left( \mu_2 - \alpha \right)^2} \mu_4^3 + 1/4 \left( \mu_2 - \alpha \right) \left[ \left( \lambda + 2 \right) \mu_2^2 + \left( \beta - \alpha \lambda - 2 \alpha \right) \mu_2 - \alpha \right] \mu_3^2 + \frac{1/4 \left( 1 + \lambda \right) \mu_3^2 - 1/2 \left( 1 - \lambda \right) \mu_2 + 1/4 \left( 1 + \lambda \right) \mu_2 + 1/4 \left( \lambda - 1 \right) \mu_2}{\eta_3 \left( 1 + \mu_2 + \mu_2 \eta_2 \right)} \eta_1 \mu_1 + 1/4 \left( \lambda - 1 \right) \mu_2 \right] \mu_1^2 + \frac{1/4 \left( 1 + \lambda \right) \mu_3^2 \left( 1 + \mu_2 \right) \lambda + \eta_2^2 \mu + \eta_2^2 \left( 1 + \mu_2 \right) \eta_1 \mu_1 + 1/4 \left( 1 + \lambda \right) \mu_1}{\eta_2 \left( 1 + \mu_2 + \mu_2 \eta_2 \right)} \mu_1^2. \right]$$  

(17)

In the remaining parts of the paper, four types of vibration transmission control are studied and compared with respect to their performance in minimising the kinetic energy index $I_k$. These are: passive control without inerter, passive control with inerter, active control without inerter and active control with inerter.
3. Passive control without inerter

In this section the effectiveness of a passive vibration isolation system without inerter is analysed as a fundamental benchmark. In this case, the dimensionless feedback gain \( \lambda \) (corresponding to the dimensional feedback gain \( g \)) and the dimensionless inerter ratio \( \mu_2 \) (corresponding to the inertance \( b_2 \)) equal zero. The transfer mobility (Eq. (11)), and the kinetic energy index (Eq. (17)) now reduce to

\[
Y_{21}(j\Omega) = \frac{2(j\Omega)^2 \eta_2 + (j\Omega)\alpha \mu_1}{(j\Omega)^4 \mu_1 + 2\eta_1(\mu_1+1)(j\Omega)^3 + \mu_1[\sigma(\mu_1+1)\beta + 1][j\Omega]^2 + 2\eta_1(1+\beta\mu_1)(j\Omega) + \mu_1[\sigma(\beta\mu_1+1) + \beta]},
\]

(18)

\[
I_k = \frac{\alpha^2 \mu_1^2(\mu_1+1) + 4\eta_1^2(\beta \mu_1+1)}{2\mu_1^2 \eta_1^2}.
\]

(19)

The effectiveness of such simple passive isolator is first studied by varying the damping ratio \( \eta_2 \) of an example system. Parameters that characterise the example system are \( \alpha = 2, \beta = 5 \) and \( \mu_1 = 1/2 \). The modulus of the transfer mobility \( |Y_{21}| \) is shown in Fig. 2.(a) for three values of the passive damping ratio \( \eta_2 \): a relatively small one (solid line), medium (dashed line) and large damping ratio (dash-dotted line).

![Fig. 2. Isolation system performance without inerter \( b_2 \) (\( \eta_2 = 0 \)): (a) Transfer mobility function \( |Y_{21}(j\Omega)| \), \( \eta_2 = \eta_{2\text{opt1}}/100 \) (solid line), \( \eta_2 = \eta_{2\text{opt1}} \) (dashed line), \( \eta_2 = 100\eta_{2\text{opt1}} \) (dash-dotted line), (b) Specific kinetic energy index \( I_k \)](image)

In case with a small damping ratio, velocity amplitudes are very large when the system is excited near either of the two natural frequencies (at dimensionless frequencies of about 1.25 and 2.75). As the damping increases, the amplitudes at these frequencies decrease. If a very large damping ratio is used, it generates a new resonance condition, at a dimensionless frequency of about 1.55. This is because for such a large damping coefficient, the dashpot between the two masses effectively locks, and two masses \( m_1 \) and \( m_2 \) vibrate together in phase, appearing to be tied rigidly. In fact, when the damping ratio \( \eta_2 \) tends to infinity, vibration amplitude at this new resonance frequency also tends to infinity since no additional damping exists in the system, \( i.e. \) undamped source and receiving structures are assumed, \( c_1 = c_3 = 0 \). A situation with
light, non-zero damping \( c_1 \) and \( c_3 \) is illustrated and discussed in the Appendix. The new, artificial natural frequency approaches

\[
\Omega_{n} = \frac{\omega_n}{\Omega} = \sqrt{\frac{1 + \beta \mu_i}{1 + \mu_i}},
\]

indicating that the system now behaves like a 1 DOF system having a natural frequency \( \omega_n = \sqrt{k/m} \) where \( k = k_1 + k_3 \) and \( m = m_1 + m_2 \). It can be seen in Fig. 2.(a) that there are four frequencies at which the mobility amplitude \( |\gamma_{21}(j\Omega)| \) is independent of the damping ratio \( \eta_2 \). By requiring that derivative with respect to \( \eta_2 \) of the modulus of the mobility \( \gamma_{21}(j\Omega) \) in Eq. (16) vanishes, the following four frequencies are obtained

\[
\Omega_{2} = 1, \quad \Omega = \sqrt{\beta}
\]

where \( |\gamma_{21}(j\Omega)| \) is invariant with respect to the isolator damping. The four circles in Fig. 2.(a) denote these frequencies and the corresponding moduli of the dimensionless transfer mobility. According to Eqs. (21a-d) and Fig. 2.(a), the entire frequency band can be divided into five ranges with respect to how vibration transmission measured through \( |\gamma_{21}(j\Omega)| \) depends on the damping ratio. As shown in Table 1., in Ranges 2 and 4 (near resonances), an increase in the damping ratio causes a decrease in the vibration transmission, but in Ranges 1, 3 and 5, the effect of increased damping is opposite. Indices a, b, c and d from Table 1. are chosen in such way that relation \( \Omega_a < \Omega_b < \Omega_c < \Omega_d \) holds for \( \beta > 1 \), whereas for \( \beta < 1 \), \( \Omega_a \) and \( \Omega_c \) now switch places.

<table>
<thead>
<tr>
<th>Range 1</th>
<th>Range 2</th>
<th>Range 3</th>
<th>Range 4</th>
<th>Range 5</th>
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</thead>
<tbody>
<tr>
<td>( \Omega \in (0, \Omega_a) )</td>
<td>( \Omega \in (\Omega_a, \Omega) )</td>
<td>( \Omega \in (\Omega, \Omega) )</td>
<td>( \Omega \in (\Omega, \Omega) )</td>
<td>( \Omega &gt; \Omega_a )</td>
</tr>
<tr>
<td>( \eta \uparrow \Rightarrow</td>
<td>\gamma_{21}(j\Omega)</td>
<td>\uparrow )</td>
<td>( \eta \uparrow \Rightarrow</td>
<td>\gamma_{21}(j\Omega)</td>
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</table>

Therefore the impact of the passive damping ratio \( \eta_2 \) at various frequency ranges is inconsistent. It is thus interesting to see how the receiving body specific kinetic energy index \( I_k \), being a frequency averaged quantity, varies with the damping ratio. This variation is plotted in Fig. 2.(b). It can be seen that it has a minimum as denoted by the circle. It can be shown that this minimum is achieved for

\[
\eta_{2\text{opt}} = \frac{\alpha \mu_i}{2 \sqrt{1 + \beta \mu_i}}.
\]

where \( \eta_{2\text{opt}} \) is the optimal passive damping ratio. The corresponding minimum specific kinetic energy is

\[
I_{k_{\text{min}}} = 2\pi \frac{\alpha \sqrt{(\beta \mu_i + 1)(\mu_i + 1)}}{\mu_i (\beta - 1)^2}.
\]

In fact, the frequency response curve indicated by the dashed line in Fig. 2.(a) corresponds to the optimal damping ratio \( \eta_{2\text{opt}} \) calculated according to Eq. (22). The remaining two lines are with a light damping
η_{2opt}/100 (solid) and a very large damping 100η_{2opt1} (dash-dotted). Thus the optimum damping ratio is a result of a trade-off between damping down the vibration transmission around the two resonances (Ranges 2 and 4) without excessively increasing vibration transmission in the remaining three frequency ranges (Ranges 1, 3 and 5).

By examining Eq. (23) it can be seen that the optimum specific kinetic energy \( I_{k\min} \) is proportional to the squared natural frequency ratio \( \alpha \), which is effectively a dimensionless measure of the isolator spring stiffness \( k_2 \). Therefore the softer the isolator spring stiffness, the better is the vibration isolation effect. However, decreasing spring stiffness \( k_2 \) normally results in large static deflections to which there is a limit in real engineering situations. In other words, opposing requirements dictate the choice of stiffness of the spring \( k_2 \). On the other hand, in Eq. (23) can be seen that if the squared natural frequency ratio \( \beta \) tends to unity, \( I_t \) tends to infinity. This is the situation in which the uncoupled natural frequency of the source body tends to the uncoupled natural frequency of the receiving body (\( \Omega_1 = \Omega_3 \)). Therefore, for a good vibration isolation effect, the system should be detuned in such way that \( \Omega_3 \approx \Omega_1 \).

In conclusion, the study of the benchmark passive isolation scheme in this section indicates that it is possible to optimise vibration isolation effects by optimising the isolator damping. However, the vibration isolation capability of such a scheme becomes very limited if \( \Omega_1 \approx \Omega_3 \). In the following section it is discussed how these limitations can be relaxed by incorporating the inerter in the isolator construction.

4. Passive control with inerter

With the inclusion of an inerter, the frequency response functions (FRF) between the receiving body motions and the source body excitation becomes characterised by an anti-resonance. This is illustrated in Fig. 3.(a) which shows the amplitude of the dimensionless transfer mobility \( |\gamma_{21}(j\Omega)| \) of an example system characterised by \( \alpha = 2, \beta = 5, \mu_1 = 1/2 \).

![Fig. 3. Transfer mobility function \( |\gamma_{21}(j\Omega)| \): (a) \( \eta_2 = 0 \) (solid line), \( \eta_2 = \eta_{2opt2} \) (dashed line), \( \eta_2 = 100\eta_{2opt2} \) (dash-dotted line), (b) \( \eta_2 = 0 \): \( \mu_2 > \alpha \) (solid line), \( \alpha/\beta < \mu_2 < \alpha \) (dash-dotted line), \( \mu_2 < \alpha/\beta \) (dashed line)](image-url)
Therefore, the system is the same as in the previous section, except that an inerter of dimensionless inertance \( \mu_2 \approx 0.692 \) is now attached to the system according to the scheme in Fig. 1. The value of \( \mu_2 \approx 0.692 \) is chosen because it is, as shown later in the paper, the optimum inertance. Active control is still switched off (\( \lambda = 0 \)). It should be noted that normally FRFs between two different locations (i.e. between two different degrees of freedom) of a linear mechanical system are characterised by at least two consecutive resonances without an anti-resonance between them, as the case is in Fig. 2.(a). Anti-resonance between all consecutive resonances can only be expected with driving point FRFs. The use of inerter changes this situation. In fact, a transfer FRF can become qualitatively similar to a typical driving point FRF, as shown in Fig. 3.(a). By inspecting the numerator in Eq. (10c), it can be seen that incorporating the inerter into the isolator places the anti-resonance at the frequency \( \omega_n = \sqrt{k_i/b_i} \), or in the dimensionless form (Eqs. (11-13))

\[
\Omega_n = \sqrt{\frac{\alpha}{\mu_2}},
\]

where index “A” denotes “anti-resonance”. Thus the location of this anti-resonance is not restricted to the frequency range between the two resonances. Instead the new zero can be freely placed in the entire frequency range assuming that the inerter with an appropriate inertance can be realised in practice. This is illustrated in Fig. 3.(b). The properties of the system in Fig. 3.(b) are still the same (\( \alpha = 2, \beta = 5, \mu_1 = 1/2 \)), except that the inertance ratio \( \mu_2 \) is varied in the range from \( \alpha/(2\beta) \) to \( 2\alpha \). In fact, it can be shown that for \( \beta > 1 \), if the inertance ratio \( \mu_2 \) is larger than the frequency ratio \( \alpha \), then the anti-resonance is positioned below the first resonance. If the inertance ratio is between \( \alpha \) and \( \alpha/\beta \), then the anti-resonance is between the two resonances. Finally, if the inertance ratio \( \mu_2 \) is smaller than \( \alpha/\beta \), then the anti-resonance is above the second resonance. This can be shown to be exactly valid for an entirely undamped system with \( \eta_2 = 0 \), and approximately valid for lightly damped systems (see Appendix to the paper). However, for \( \beta < 1 \) and if the inertance ratio \( \mu_2 \) is smaller than the frequency ratio \( \alpha \), then it can be shown that anti-resonance is positioned above the second resonance and if the inertance ratio \( \mu_2 \) is larger than \( \alpha/\beta \), then the anti-resonance is below the first resonance. This free choice of the anti-resonance opens possibilities to control “tonal” vibration transmission due to simple harmonic excitation. In this case, a lightly damped isolator would be necessary for a maximised performance which is perhaps similar to how vibration neutralisers (lightly damped, single frequency tuned vibration absorbers) are designed [49], [50].

However, in this study the focus is put onto a broadband, white noise dynamic excitation of the source body. It is shown next that the kinetic energy index of the receiving body \( I_t \) has a minimum with respect to both damping ratio \( \eta_2 \) and inertance ratio \( \mu_2 \). Differentiating Eq. (17) with respect to \( \eta_2 \) yields

\[
\frac{\partial I_t}{\partial \eta_2} = 4\pi \left\{ -1/4 \mu_2 (\mu_2 \beta - \alpha^2) \mu_2^4 - 1/4 (\mu_2 \beta - \alpha) \left[ 2 \mu_2^2 + (\beta - 2 \alpha) \mu_2 - \alpha \right] \mu_2^3 + \right. \\
+ \left. \left[ (1 + \beta) \mu_2 + \beta \right] \eta_2^2 \mu_2 + \eta_2^2 (1 + \mu_2) \right\}.
\]

By equalling Eq. (25) to zero and solving for \( \eta_2 \), one obtains
\[ \eta_2 = \frac{\mu_1}{2} - \frac{\sqrt{(1 + \beta \mu_1)^2 \mu_2^2 + \left[-2(\mu_1 + 1)(1 + \beta \mu_1)\alpha + 1 + \mu_2 \beta^2\right] \mu_2^2 - 2\alpha \left[-1/2(\mu_1 + 1)^2 \alpha + \beta \mu_1 + 1\right] \mu_2 + \alpha^2 (\mu_1 + 1)}}{(1 + \mu_2 + \mu_1 \mu_2)(1 + \beta \mu_1)}, \]  

(26)

where \( \eta_2 \) is now the optimal damping ratio for a given inertance ratio \( \mu_2 \). By substituting Eq. (26) into Eq. (17), differentiating with respect to \( \mu_2 \), equalling with zero and solving for \( \mu_2 \), optimal inertance ratio parameter \( \mu_{2\text{opt}} \) may be obtained which subsequently yields an expression for a minimum specific kinetic energy \( I_k \). However, explicit expressions for \( \mu_{2\text{opt}}, \eta_{2\text{opt}} \) and \( I_{k\text{opt}} \) are omitted in this paper because they are rather involved and cannot be clearly interpreted. Nevertheless, Fig. 4.(a) shows that the frequency averaged kinetic energy index of the receiving body has a minimum at \( \eta_2 = \eta_{2\text{opt}} \) and \( \mu_2 = \mu_{2\text{opt}} \). The example system considered is still characterised by \( \alpha = 1/2 \), \( \beta = 5 \) and \( \mu_1 = 1/2 \). The optimum values for this system are \( \eta_{2\text{opt}} \approx 0.162 \) and \( \mu_{2\text{opt}} \approx 0.692 \). The minimum position for \( I_k \) is denoted by the circle whereas the optimum damping ratio \( \eta_2 \) as a function of the inertance ratio \( \mu_2 \) (Eq. (26)) is illustrated using the dashed line. Considering now again Fig. 3.(a), the effects of varying the damping ratio \( \eta_2 \) are similar to the case without inerter. With light damping, the receiving body response is large around the two resonances, and with a damping coefficient too large, a new lightly damped resonance occurs as with the case in previous section with no inerter. However, under the optimal setting, the vibration velocity amplitude is further reduced in comparison to the case without inerter (\( \mu_2 = 0 \)). This is due to the anti-resonance phenomenon which is a special case of inerter influence and otherwise cannot be obtained by combining only elements of classical mass-damper-spring (MDS) system if the scheme shown in Fig. 1. is followed. Similar to the case without inerter, it can be seen in Fig. 3.(b) that there are four frequencies where the dimensionless transfer mobility amplitudes \( |Y_12(j\Omega)| \) are independent of damping ratio \( \eta_2 \). The four circles in Fig. 3.(a) denote these frequencies and the corresponding dimensionless mobility amplitudes \( |Y_{11}(j\Omega)| \). They can be calculated as

\[ \Omega_1 = 1, \quad \Omega_2 = \sqrt{\beta} \]

\[ \Omega_{a,c} = \sqrt{\frac{2(1 + 2\mu_2 + 2\mu_2 \mu_1)}{2 + 4(\mu_1 + 1)\mu_1}} \]

(27a-d)

\[ \Omega_{b,c} = \sqrt{\frac{2(1 + 2\mu_2 + 2\mu_2 \mu_1)}{2 + 4(\mu_1 + 1)\mu_1}} \]

It can be noted that expressions (27a,b) are identical to (21a,b) for the case without inerter. However, with inerter, the values of \( \Omega_{a,b,c,d} \) can now all switch places depending on how the parameters of the system are chosen, except that \( \Omega_{b,c} \) from Eq. (27c-d) are always such that inequality \( \Omega_b < \Omega_c \) holds.
Fig. 4. Isolation system performance with inerter $b_2$ ($\mu_2 = \mu_{2\text{opt}}$): (a) specific kinetic energy index $I_k$ as a function of $\eta_2$ and $\mu_2$, (b) specific kinetic energy index $I_k$: comparison of the cases with and without inerter; $\mu_2 = 0$ (solid line), $\mu_2 = \mu_{2\text{opt}}$ (dashed line).

It is interesting to compare the maximum vibration isolation effect obtained using an optimally tuned damper $c_2$ without inerter ($\eta_{2\text{opt1}}$) to that with an optimally tuned damper and inerter pair ($\mu_2 = \mu_{2\text{opt}}$, $\eta_{2\text{opt2}}$). This comparison is shown in Fig. 4.(b). The plot shows the receiving body specific kinetic energy index $I_k$ as a function of the passive damping ratio $\eta_2$ for the two cases. It can be seen that the use of inerter improves the optimised vibration isolation performance by ~3 dB for the case considered. Clearly, this improvement depends on the parameters that characterise the system. Fig. 5. shows the improvement designated in Fig. 4.(b) as a function of the system parameters $\alpha$, $\beta$ and $\mu_1$.

Fig. 5. Minimum specific kinetic energy ratio index $I_{k\text{min,R}}$, $\lambda = 0$: (a) $\alpha = 1/2$ (solid line), $\alpha = 10$ (dashed line), $\alpha = 10^3$ (dash-dotted line), (b) $\mu_1 = 1/2$ (solid line), $\mu_1 = 10$ (dashed line), $\mu_1 = 10^3$ (dash-dotted line)

The parametric study is presented so that the ratio between the optimum kinetic energy index with inerter and the optimum kinetic energy without inerter is shown as a function of the frequency parameter $\beta$. In Fig.
5. (a) this variation is shown for three different frequency ratios $\alpha$, and in Fig. 5. (b) the variation is shown for three different parameters $\mu_1$. The fixed parameter of the example systems shown in Fig. 5. (a) is $\mu_1 = 1/2$, while the fixed parameter of the example systems shown in Fig. 5. (b) is $\alpha = 1/2$. It is apparent from the two plots that the best improvements due to the use of the inerter are obtained if the source and receiving bodies have similar uncoupled resonance frequencies, that is if $\beta$ approaches unity, or $\Omega_1 \approx \Omega_3$. However, these results must be interpreted with care. This is because if the two resonance frequencies are exactly the same, then both configurations with and without inerter are equally ineffective, as shown by the vertical lines in the two plots which tend to 0 dB if $\beta$ is exactly one. This is due to the fact that $c_1 = c_3 = 0$ is assumed. In Appendix to the paper, a plot analogue to the two plots in Fig. 5. is shown, in which results obtained by considering various levels of passive damping $c_1$ and $c_3$ were considered. The plot in the Appendix indicates that for systems with non-negligible, but still light damping $c_1$ and/or $c_3$, the use of inerter gives the best vibration isolation improvement at the frequency ratio $\beta$ slightly below and slightly above unity. With large damping coefficients $c_1$ and $c_3$, the best effects of including the inerter are again observed with $\beta = 1$.

In conclusion, the passive isolator scheme enhanced by incorporating an inerter, exhibits an improved broadband vibration isolation performance in terms of the specific kinetic energy of the receiving body. In the following section it is investigated how much can the vibration isolation performance be further increased by engaging the velocity feedback unit in the isolator scheme shown in Fig. 1.

5. Active control

5.1. Stability in general

With the frequency domain analysis, the stability of active control systems cannot be seen directly from the frequency response of the system. In other words, the model presented in section 2 mathematically allows for calculating frequency response functions using Eq. (10) for both stable and unstable systems. However, such FRFs for unstable systems would be physically meaningless. It is thus necessary to carefully investigate the active control system stability properties before calculating the prospective performance metrics, such as the kinetic energy index given by Eq. (16). It has previously been shown that active vibration isolation systems can exhibit stability problems as discussed in for example [29], [43], [51], [52]. In this subsection, the stability of the feedback control loop is studied with reference to the dimensionless active damping coefficient $\lambda$ by applying the Routh-Hurwitz [53], [54] stability criterion to the characteristic equation of the system. The characteristic equation is the denominator of Eq. (11).

According to the Routh-Hurwitz necessary stability condition and Eq. (12) in order for $A_{1,3} > 0$, it must be

$$\forall \beta < 1 \Rightarrow \lambda > -(\beta \mu_1 + 1) \text{ in order for } A_1 > 0,$$

$$\forall \beta > 1 \Rightarrow \lambda > -(\mu_1 + 1) \text{ in order for } A_3 > 0.$$
In other words, if $\beta < 1$ the condition $A_1 > 0$ is a stricter one and if $\beta > 1$, then $A_3 > 0$ is the stricter condition. Considering now the Routh-Hurwitz sufficient condition for stability, it states that all diagonal subdeterminants $H_1$, $H_2$ and $H_3$, as well as the main determinant $H_4$ of Hurwitz matrix must be positive. The principal determinant $H_4$ is proportional to the sub-determinant $H_3$ with an always positive term $\mu_i(\alpha(\beta\mu_i + 1) + \beta)$ and is thus automatically positive if $H_3$ is positive. Thus the relevant criteria that must be satisfied simultaneously are Eqs. (28) and (29) plus the following additional ones

$$H_1 > 0 \Rightarrow \lambda + \mu_i + 1 > 0.$$  \hspace{1cm} (30)
$$H_2 > 0 \Rightarrow 2\mu_i \eta_i \mu^2 \mu_i \left( \mu_i (\beta - 1) + \alpha \right) + \mu_i \lambda + \alpha + \beta > 0.$$  \hspace{1cm} (31)
$$H_3 > 0 \Rightarrow 4\mu_i^2 \eta_i^2 (\beta - 1) \left( (\mu_i - \alpha) \lambda^2 + \beta \mu_i (\mu_i + 1) \lambda + \beta \lambda - 1 \right) > 0.$$  \hspace{1cm} (32)

Note that $A_1$, $A_3$, $H_1$ and $H_2$ are linear functions of the dimensionless feedback gain $\lambda$, whereas $H_3$ is a quadratic function of $\lambda$. The quadratic determinant $H_3$ changes sign at the following values of the feedback gain

$$\lambda_{1,2} = \frac{\mu_i (\beta \mu_i + 1) - \alpha (\mu_i + 1) + \beta - 1 \pm \sqrt{(\alpha - \mu_i \beta)^2 \mu_i^2 + 2 \mu_i (\alpha - \mu_i \beta)(\alpha + 1 - \mu_i - \beta) + (\alpha - \mu_i + \beta - 1)^2}}{2(\alpha - \mu_i)}.$$  \hspace{1cm} (33a,b)

In the forthcoming discussion, it is shown that by ensuring the validity of inequality (32), all other stability conditions are satisfied automatically. In other words, the condition (32) is a sufficient stability condition for the problem studied.

5.1.1. Stability without inerter – subcritical and supercritical systems

If the inerter is not used, i.e. $\mu_2 = 0$, Eqs. (33a,b) can be simplified to

$$\lambda_{1,2} = \frac{\beta - 1 - \alpha (\mu_i + 1) \pm \sqrt{(1 + \mu_i)^2 \alpha^2 - 2 (\mu_i - 1)(\beta - 1)\alpha + (\beta - 1)^2}}{2\alpha},$$  \hspace{1cm} (34a,b)

where $\lambda_1$ is the lower value out of two zeros. At this point it is convenient to graphically represent all expressions relevant for the system stability as a function of the dimensionless feedback gain $\lambda$. This is done in Fig. 6. Two different cases are presented. Fig. 6.(a) shows the case when the squared natural frequency ratio $\beta < 1$ and Fig. 6.(b) shows the case with $\beta > 1$. The parameters for an example system shown in Fig. 6.(a) are $\alpha = 2$, $\mu_i = 1/2$, $\eta_i = 1$ and $\beta_{(a)} = 1/2$, and the parameters for an example system shown in Fig. 6.(b) are the same, except $\beta_{(b)} = 5$. The zeros of the 3rd principal diagonal minor $H_3$ from Eqs. (34a,b) are denoted by the two circles. In both plots can be seen that if the principal diagonal minor with the quadratic dependence on the feedback gain is positive, i.e. $H_3 > 0$, then all other stability conditions are automatically satisfied. In fact, by closely inspecting Eqs. (28-32) it could be deduced that it is generally true that if $H_3 > 0$ then all other conditions, i.e. Eqs. (28-31), are automatically satisfied and the stability is guaranteed. Thus Eq. (32) represents the strictest stability condition and it becomes sufficient to make sure that $H_3 > 0$ in order to have a stable feedback loop.
Physically this indicates that if the uncoupled natural frequency of the source body is larger than the uncoupled natural frequency of the receiving body, then a negative velocity feedback loop with an arbitrary large feedback gain can be used. As discussed in the forthcoming section 5.2, this is a situation in which very convincing active vibration isolation effects can be achieved that can significantly outperform the two passive vibration isolation schemes discussed in the previous two subsections. On the other hand, in situation in which the uncoupled natural frequency of the source body is smaller than the uncoupled natural frequency of the receiving body, as shown in Fig. 6.(b), the range of dimensionless feedback gains is limited between $\lambda_1$ and $\lambda_2$, according to Eq. (34), and shown in Fig. 6.(b). Therefore, the maximum feedback gain is limited by $\lambda_2$ above which the second order principal diagonal minor becomes negative with further increasing the feedback gain. This is because the parabola in Fig. 6.(b) is oriented downwards whereas the parabola in Fig. 6.(a) is oriented upwards. This situation results in limited active vibration control performance as discussed in the forthcoming section 5.2.

In conclusion, it can be stated that all systems representable by the scheme in Fig. 1. can be divided into two families. The first family can be referred to as supercritical and it is characterised by $\beta < 1$. The systems belonging to this group allow for the implementation of unconditionally stable active vibration isolation scheme based on the direct feedback of the absolute velocity of the receiving body. The second family is characterised by $\beta > 1$ and it can be referred to as subcritical. The systems belonging to this group do not allow for the implementation of unconditionally stable absolute velocity feedback scheme. On the contrary, the feedback loop is conditionally stable with a limited maximum feedback gain.

Practical vibration isolation problems belonging to the supercritical family are the problems of isolating vibrations coming from a flexible base towards sensitive equipment mounted on the base. A practical problem belonging to the subcritical group could be a problem in which running machinery is elastically mounted on the flexible base, for example a punching press. In such case, the broadband vibrations
originating from the impact, transmit from the machine to the base. It appears from the above analysis that it would be significantly more difficult to guarantee the stability of the absolute velocity feedback control applied on the latter, subcritical family of vibration isolation problems. Given these difficulties, it is interesting to investigate the effects of the use of an inerter with subcritical systems characterised by \( \beta > 1 \). This investigation is carried out in the following subsection.

5.1.2. Stability with inerter – stabilising the feedback loop in a subcritical system

If an inerter is used in an isolator of a subcritical system characterised by \( \beta > 1 \), then interesting effects can be observed with regard to the stability of the feedback loop. By inspecting Eq. (32), it can be seen that the third principal diagonal minor \( H_3 \), which is essential for the stability of the active control, has the quadratic coefficient in \( \lambda \) equal to \( \mu_2 - \alpha \). This coefficient determines whether the corresponding parabola is pointing upwards or downwards. Given that the term \( (\beta - 1) \) multiplying the squared bracket expression is positive with subcritical systems, it turns out that an inerter with dimensionless inertance \( \mu_2 > \alpha \) can make the quadratic coefficient of the parabola positive. This in turn results in an upward pointing parabola. Therefore unconditional stability can be achieved also with subcritical systems simply by adding an inerter with \( \mu_2 > \alpha \). This is illustrated in Fig. 7(a) which shows all principal diagonal minors calculated according to Eqs. (30 -32). The system is again characterised by \( \alpha = 2, \mu_1 = 1/2, \eta_2 = 1 \) and \( \beta = 5 \), just like in Fig. 6(b). As shown in Fig. 7(a), with the inclusion of inerter when \( \beta_0 > 1 \) and \( \mu_2(\alpha) = \alpha/2 \), the limited stable range of \( \lambda \) between \( \lambda_{H1} < \lambda < \lambda_{H2} \), is expanded in comparison with Fig. 6(b). If the inertance is further increased, so that \( \mu_2 = 2\alpha \), the system becomes stable for any \( \lambda > \lambda_{H2} \), as shown in Fig. 7(b). Therefore, for subcritical systems where the fundamental natural frequency of the receiving body is larger than that of the source body, the use of inerter characterised by \( \mu_2 > \alpha \) drastically improves the stability by turning a subcritical active vibration isolation problem into a supercritical one. This is quite essential for the performance of the active vibration isolation, as discussed in the following subsections.

Fig. 7. Dependency of Hurwitz coefficients \( H_1 \) (solid line), \( H_2 \) (dashed line), \( H_3 \) (dash-dotted line) and \( A_1 \) (dotted line) magnitude on active damping ratio \( \lambda \) with inerter \( b_2 (\mu_2 \neq 0) \) and \( \beta_0 > 1 \): (a) \( \mu_2 < \alpha \), (b) \( \mu_2 > \alpha \)
5.2. Performance

5.2.1. Without inerter

The performance of the active control is first studied without the use of inerter, therefore dimensionless parameter $\mu_2$ equals zero. Fig. 8.(a) shows the specific kinetic energy index of the receiving body plotted as a function of the passive and active damping ratio. Firstly, a supercritical system is assumed so the frequency ratio $\beta$ is smaller than one. Fig. 8.(a) indicates that as the active damping ratio (the feedback gain) is increased, the kinetic energy index monotonically decreases demonstrating that the desired active vibration isolation effect is achieved. Fig. 8.(b) shows the dimensionless transfer mobility function (the velocity of the receiving body per unit forcing of the source body, as a function of frequency) for increasing active damping ratios.

![Fig. 8. Isolation system performance without inerter $b_2$ ($\mu_2 = 0$) and $0 < \beta_i < 1$: (a) Specific kinetic energy index $I_k$, (b) Transfer mobility function $|\gamma_{21}^\ast (j\Omega)|$, $\lambda = 0$ (solid line), $\lambda = 2$ (dashed line), $\lambda = 10$ (dash-dotted line), $\lambda = 20$ (dotted line)](image)

It can be seen that the amplitude of the dimensionless mobility function $|\gamma_{21}^\ast (j\Omega)|$ diminishes in the vicinity of $\Omega_{n1}$ and $\Omega_{n2}$ which is tied to significant reduction of specific kinetic energy $I_k$. In addition, no increase of the amplitude of the mobility with an increase in the feedback gain can be seen at any frequency. Thus a true broadband active vibration isolation effect can be achieved. The characteristic parameters of the example system illustrated in Fig. 8.(a) are $\alpha = 1/2$, $\beta = 1/2$ and $\mu_1 = 1/2$. Parameters of the example system shown in Fig. 8.(b) are the same, except that the damping ratio had to be fixed to $\eta_2 = 0.5 \%$. Therefore in such supercritical system, the use of inerter appears to be unnecessary, since the system is stable for any given positive value of $\lambda$.

Considering now the subcritical case, where $\beta_{ii} > 1$, the system is stable for a limited narrow $\lambda$ range as already discussed in section 5.1 and as shown in Fig. 6.(b). Therefore it is interesting to investigate into
performance of the active control for subcritical systems when the stable feedback gain is restricted between the lower and upper margins shown in Fig. 6.(b).

The kinetic energy index of the receiving body in this case $(\beta_{II} > 1)$ is shown in Fig. 9.(a). The parameters of the example system shown in Fig. 9.(a) are $\alpha = 1/2$, $\beta = 2$ and $\mu_1 = 1/2$.

It can be seen in Fig. 9.(a) that there is an optimum combination of the passive and active damping ratios that minimises the kinetic energy index which is marked by the red circle. The optimum passive damping ratio as a function of the active damping ratio is shown by the red dashed line in Fig. 9.(a). Similar to Eq. (22), this function is obtained by differentiating Eq. (17) with respect to $\lambda$ and equalling with zero which yields the following relation

$$\eta_{2opt,\lambda} = \frac{\alpha \mu_1}{2} \frac{1 + \mu_1 + \lambda}{1 + \beta \mu_1 + \lambda}. \quad (35)$$

By comparing it with Eq. (22), it can be observed that $\lambda$ is now added under the root of both numerator and denominator. If $\lambda = 0$, Eq. (35) reduces to Eq. (22). Eq. (35) is denoted by dashed line in Fig. 9.(a). By inserting Eq. (35) into Eq. (17), an expression for minimum specific kinetic energy along the dashed line can be obtained

$$I_{kmin} = 2\pi \frac{\alpha \sqrt{(\beta \mu_1 + \lambda + 1)(\mu_1 + \lambda + 1)}}{\mu_1 (\beta - 1) [-1 + \beta - \mu_1 \alpha - \alpha] \lambda - 1 + \beta - \alpha \lambda^2]. \quad (36)$$

By differentiating Eq. (36) with respect to $\lambda$, equalling with zero and solving for $\lambda$, optimal active damping coefficient $\lambda_{opt}$ is obtained. Inserting both $\eta_{2opt,\lambda}$ from Eq. (35) and $\lambda_{opt}$ into Eq. (17) yields an expression for minimum specific kinetic energy $I_k$. However, the relations for $\lambda_{opt}$, $\eta_{2opt,\lambda}$ and $I_k$ are too cumbersome and not easily interpretable, so they are omitted. Nevertheless the global minimum position for $I_k$ with respect to two variables $\lambda_{opt}$ and $\eta_{2opt,\lambda}$ exists and it is denoted by the circle in contour plot. The asterisk in Fig. 9.(a) denotes the minimum kinetic energy index if value of $\lambda$ is set to zero, which implies the use of optimised
passive control. By comparing the surface levels in Fig. 9.(a) at the optimum active control (circle) and the optimum passive control (asterisk) it can be seen that the level difference is about one dB. Therefore, the active control can outperform the passive control, but the corresponding improvement in performance is not particularly convincing. It can be concluded that with subcritical system the performance of the active vibration scheme is questionable, since a significantly simpler passive system can achieve nearly the same vibration isolation effect. The reasons for this are further investigated by plotting the dimensionless transfer mobility \(|R_{21}(j\omega)|\) as a function of frequency for cases with no control (\(\lambda = 0\)) and with the active control using increasing active damping ratios (increasing feedback gains) in Fig. 9.(b). The parameters for the example system shown in Fig. 9.(b) are the same as in Fig. 9.(a), except that a fixed passive damping ratio \(\eta_2 = 0.02\) is used. It can be seen in Fig. 9.(b), that although the amplitude of the dimensionless mobility \(|R_{21}(j\omega)|\) reduces in the vicinity of second dimensionless circular frequency \(\Omega_{n2}\) with rising \(\lambda\), a significant overshoot can be observed in the vicinity of first dimensionless circular frequency \(\Omega_{n1}\) for rising \(\lambda\). Fig. 10.(a) shows that for rising \(\lambda\), the specific kinetic energy \(I_k\) also rises significantly until instability occurs. Therefore, using active control without inerter in cases when \(\beta_{II} > 1\) results in generally doubtful performance. Fig. 10.(b) shows the comparison between the optimum active vibration isolation and the optimum passive vibration isolation for a subcritical system characterised by \(\alpha = 1/2\), \(\beta = 2\) and \(\mu_1 = 1/2\) in terms of the amplitude of the transfer mobility \(|R_{21}(j\omega)|\) plotted as a function of frequency. The same parameters are used in Fig. 10.(a) where the passive damping ratio is set to \(\eta_2 = 0.02\). The optimised active control results in a slightly lesser kinetic energy index compared to the optimised passive control (Fig. 10.(a)), which is obtained by damping down the velocity response around the first natural frequency at the expense of slightly increasing the response around the second natural frequency (Fig. 10.(b)). In conclusion, the improvement in the performance due to the use of active control probably does not justify the complexity of the vibration isolation system in subcritical systems.

![Graph](image.png)

**Fig. 10.** Isolation system performance without inerter \(b_2 (\mu_2 = 0)\) and \(\beta_{II} > 1\): (a) Specific kinetic energy index \(I_k\), (b) Transfer mobility function \(|R_{21}(j\omega)|\), \(\lambda = 0, \eta_2 = \eta_{2opt1}\) (solid line), \(\lambda = \lambda_{opt}, \eta_2 = \eta_{2opt,\lambda}\) (dashed line)
5.2.2. Comment on the reciprocity principle

So far, it has been shown that all active vibration isolation problems shown in Fig. 1, can be categorised as subcritical or supercritical, depending on the frequency ratio $\beta$. The above stability and performance analyses indicate that with supercritical systems, the active vibration isolation based on absolute velocity feedback can be expected to yield very good vibration isolation effects. With subcritical systems this is not the case due to poor stability margins and the consequent unconvincing vibration isolation performance.

At this point, it should be noted that this division into supercritical/subcritical systems is based on the assumption that the velocity sensor is mounted on the receiving body. With respect to that, one possible way of turning a subcritical vibration isolation problem into a supercritical one would be to place the sensor on the source body (see Fig. 1, - dashed alternative feedback loop). Then, the body where the sensor is located is characterised by the larger natural frequency than the other body, and the new feedback loop will be stable for an arbitrary large gain implementing a negative velocity feedback since everything else in the system is symmetric. In fact, the transfer mobility $Y_{12}(j\Omega)$, in the case the sensor is located on the source body will be affected by the control loop in the same way the transfer mobility $Y_{21}(j\Omega)$ is affected by the control loop if the sensor is mounted on the receiving body. Similarly, the transfer mobility $Y_{21}(j\Omega)$, in the case the sensor is located on the source body, will be affected by the control loop in the same way $Y_{12}(j\Omega)$ is affected by the control loop if the sensor is mounted on the receiving body. In other words, if the sensor is located on the source body of a system characterised by $\Omega_3 > \Omega_1$ (subcritical), the velocity of the mass $m_1$ due to the force exciting the mass $m_2$ should decrease with increasing the feedback gain similarly to what is shown in Fig. 8.(b). The question however is what happens with the velocity of the mass $m_2$ due to forcing at the mass $m_1$ in such case. According to the reciprocity principle which states that $Y_{21} = Y_{12}$, the velocity of the mass $m_2$ should monotonically decrease with the increase of the feedback gain and this should provide a simple solution to the vibration isolation problem in subcritical systems.

However, the reciprocity principle does not hold if the system is made active. In other words, $Y_{21} \neq Y_{12}$ if $\lambda \neq 0$ which can be seen by comparing Eqs. (10b) and (10c). Although the two frequency response functions are characterised by the same denominators, they have different numerators. In fact, the two numerators become the same only if the feedback gain is zero. This is a consequence of the diagonally asymmetric damping matrix from Eq. (4b). The disruption of the reciprocity principle is illustrated on an example system in Fig. 11.
Disruption of the reciprocity principle, $\mu_2 = 0$, transfer mobility function $y_{ij}(j\Omega)$. $y_{12} = y_{21}, \lambda = 0$ (dotted line), $y_{12}, \lambda = 25$ (solid line), $y_{21}, \lambda = 25$ (dashed line).

The example system parameters are $\alpha = 1, \beta = 1/2, \mu_1 = 1$ and $\eta_2 = 0.02$. The dimensionless feedback gain $\lambda$ assumes either a very large ($\lambda = 25$), or zero value ($\lambda = 0$). Fig. 11 shows a clear difference between $y_{21}$ (dashed line) and $y_{12}$ (solid line) in case the active damping ratio of $\lambda = 25$ is used. Only if the active damping ratio is set to zero, then the two mobilities become the same as indicated by the red dotted line in Fig. 11.

This prevents a straightforward solution of the subcritical vibration isolation problem by placing the sensor on the source body. Although this would solve the stability problem, it would not improve the isolation performance. This is because, as shown by the solid line, the mobility function $y_{12}$ becomes characterised by a significant overshoot at higher frequencies.

However, as discussed in the forthcoming part of the paper, the use on an inerter in the isolator may be a viable solution to the subcritical vibration isolation problems. It has already been shown in section 5.1.2 that the use of inerter with inertance $\mu_2 > \alpha$ opens one of the two stability margins and enables the use of large, theoretically unlimited feedback gains in subcritical systems. In the following subsection the performance of such an inerter-based active vibration isolation system is discussed, i.e. it is shown how such control affects the kinetic energy of the receiving body.

5.2.3. With inerter

Fig. 12.(a) shows the specific kinetic energy $I_k$ plotted as a function of the active damping ratio $\lambda$ of a subcritical system characterised by $\alpha = 1/2, \beta = 2, \mu_1 = 1/2$ and $\eta_2 = 0.01$, equipped with an inerter of inertance $\mu_2 = 2$. Therefore an inertance large enough to stabilise the feedback loop is used ($\mu_2 > \alpha$). It can be seen that with an increase in the dimensionless feedback gain $\lambda$, the specific kinetic energy index monotonically decreases indicating that the desired vibration isolation effect is accomplished. Fig. 12.(b) shows the amplitude of the dimensionless transfer mobility $|y_{21}(j\Omega)|$ plotted against frequency for increasing feedback gains. Note the anti-resonance effect at frequencies below the first resonance, introduced by inerter. It can
be seen that with the increase in the feedback gain, the receiving body response is decreased at both resonance frequencies. The higher the gain, the lower is the velocity response. There can be seen no frequencies at which the increase of the feedback gain causes an increase in the response. Therefore, it can be concluded that the inclusion of the inerter in the active vibration isolation scheme with subcritical problems is essential in establishing stable and efficient active vibration isolation. It should be noted that the inerter can be seen from the control point of view as a relative acceleration feedback. In other words, subtracted outputs of two accelerometers mounted on the receiving and source bodies could theoretically be fed to the reactive actuator in addition to the existing velocity feedback in order to synthesize the inerter element actively. However, such “derivative” active vibration control has never been achieved in practice to the best of authors’ knowledge. It appears that the corresponding sensor-actuator frequency response function does not roll-off with frequency which causes very pronounced stability problems associated with high frequency poles, as discussed for example in [55]. It is therefore very useful in the scheme to include the inerter as a passive element which mimics the effects of a relative acceleration feedback to reactive force actuator.

![Fig. 12. Isolation system performance with inerter $b_2 (\mu_2 \neq 0), \beta_0 > 1$ and $\mu_2 > \alpha$: (a) Specific kinetic energy index $I_k$, (b) Transfer mobility function $|\mathcal{Y}_{21}(j\omega)|$; $\lambda = 0$ (solid line), $\lambda = 5$ (dashed line), $\lambda = 10$ (dash-dotted line), $\lambda = 20$ (dotted line)](image)

### 6. Conclusions

In this paper, a novel, inerter-based active vibration isolation system is presented. Two fundamental passive benchmark isolators are also investigated, one not employing the inerter and the other employing the inerter. The methodology is studied on a simple two degree of freedom system so that many conclusions can be drawn based on analytically derived expressions. Such a simplified system can be seen as a reduced order model of a potentially more complex structure. It is shown in the paper that the vibration isolation performance of the fundamental passive isolator not employing the inerter can be improved by adding the inerter in parallel with the isolator spring and damper. This improvement is particularly significant if the
source and receiving bodies have similar uncoupled natural frequencies. By investigating the stability of the active control when no inerter is used, it is found that there are two fundamental families of vibration isolation problems. With the first family (supercritical systems), which is characterised by the natural frequency of the uncoupled source body larger than the natural frequency of the uncoupled receiving body, large feedback gains can be used without compromising the stability of the feedback control system. This results in a convincing broadband vibration isolation effect. With the second family of systems (subcritical systems), the natural frequency of the uncoupled source body is below the natural frequency of the uncoupled receiving body. The range of stable feedback gains is limited which results in poor vibration isolation performance. However with the inclusion of the inerter, broadband active vibration isolation can also be achieved in the subcritical family of systems. Adding the inerter into the isolator effectively generates a sort of relative acceleration feedback that stabilises the control loop. In fact, it is analytically calculated in the paper that the minimum inertance to stabilise the loop is proportional to the stiffness of the isolator spring and inversely proportional to the squared natural frequency of the source body. It is important to mention that direct acceleration feedback would not be possible in practice due to very pronounced stability problems, therefore the passive element which mimics such feedback is very useful.

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**Appendix A**

So far it has been assumed that the passive damping of the source and the receiving body is light to the limit that it becomes negligible and the corresponding damper coefficients \(c_1\) and \(c_3\) were assumed to be zero. In this Appendix, the validity of these assumptions is tested. This is done by plotting some characteristic results in case when the damping coefficients \(c_1\) and \(c_3\) are set so that the two corresponding damping ratios \(\eta_{1,3}\) both equal 1 %. The corresponding damping ratios are defined as

\[
\eta_1 = \frac{c_1}{2\sqrt{m_1k_1}} , \quad \eta_3 = \frac{c_3}{2\sqrt{m_3k_3}} .
\]

The first situation that is discussed is with reference to Fig. 2, where the damping coefficients \(c_1\) and \(c_3\) were neglected. In Fig. A1., the plots analogue to those of Fig. 2. are shown with \(\eta_1 = \eta_3 = 0.01\).
Fig. A1. Isolation system performance without inerter $b_2 (\mu_2 = 0)$ for $\eta_1 = \eta_3 = 0.01$: (a) Transfer mobility function $|Y_{21}(j\Omega)|$, $\eta_2 = \eta_{2\text{opt}1}/100$ (solid line), $\eta_2 = \eta_{2\text{opt}1}$ (dashed line), $\eta_2 = 100\eta_{2\text{opt}1}$ (dash-dotted line), (b) Specific kinetic energy index $I_k$

All other parameters that characterise the system are the same as in Fig. 2. It can be seen that the qualitative response of the system is the same as that illustrated in Fig. 2. However, quantitatively the minimum kinetic energy in Fig A1.(b) is about 1/2 dB lower than in Fig. 2. Also, the corresponding optimum damping ratio $\eta_2$ is slightly higher than in the situation illustrated in Fig. 2. Therefore it can be stated that the basic conclusions drawn for undamped source and receiving bodies, are also valid for lightly damped source and receiving bodies. The analytical simplified expressions for the optimum damping of the isolator $\eta_2$ and the minimum kinetic energy index are also approximately valid under the assumption of lightly damped source and receiver bodies. Considering now the passive vibration isolation with the use of inerter, Fig. A2. depicts results corresponding to Fig. 3. when the damping $\eta_1 = \eta_3 = 0.01$. Again, all other parameters that characterise the system are the same as in Fig. 2. In Fig. A2., it can be seen that the same qualitative characteristics of the receiving body response are observed with lightly damped source and receiving bodies in case ineters are incorporated in the passive isolator.

Fig. A2. Transfer mobility function $|Y_{21}(j\Omega)|$ for $\eta_1 = \eta_3 = 0.01$: (a) $\eta_2 = 0$ (solid line), $\eta_2 = \eta_{2\text{opt}2}$ (dashed line), $\eta_2 = 100\eta_{2\text{opt}2}$ (dash-dotted line), (b) $\eta_2 = 0$: $\mu_2 > \alpha$ (solid line), $\alpha/\beta < \mu_2 < \alpha$ (dash-dotted line), $\mu_2 < \alpha/\beta$ (dashed line)
The following figure, Fig. A3, aims to illustrate the influence of the light damping ratios $\eta_1 = \eta_3 = 0.01$ on the optimum combination of inertance and damping ratio $\eta_2$ in Fig. A3.(a), and on the improvement of the vibration isolation performance due to the added inerter. It can also be seen in Fig. A3.(a) that the optimum combination the inertance ratio $\mu_2$ and damping ratio $\eta_2$ is virtually not affected. Fig. A3.(b) on the other hand shows that with lightly damped source and the receiving bodies, the kinetic energy of vibration of the receiving body reduces both with and without inerter. The relative improvement due to the inerter however remains to be $\sim 3$ dB.

A plot analogue to the two plots in Fig. 5. is shown next in Fig. A4., where results obtained by considering various levels of passive damping $c_1$ and $c_3$ are considered. It can be seen that as the damping coefficients $c_1$ and $c_3$ (and the corresponding damping ratios $\eta_1$ and $\eta_3$) are reduced, the plot becomes similar to the plots in Fig. 5. (i.e. the red solid line in Fig. A4.). However, as the two damping coefficients approach about 0.1 \%, the plots cease to resemble those in Fig. 5. which are valid for undamped source and receiving bodies.
This is shown by the magenta dotted line and the green thick solid line in Fig. A4. In such cases with low to moderate damping ratios of the two bodies, it appears that the greatest improvement of the vibration isolation performance by adding the inerter in the isolator scheme can be expected with source and receiving bodies having similar uncoupled natural frequencies ($\beta \approx 1$).

References


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