Vibration analysis of rotating toroidal shell by the Rayleigh-Ritz method and Fourier series

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Abstract

In this self-contained paper free vibrations of a pressurised toroidal shell, rotating around its axis of symmetry, are considered. Extensional and bending strain-displacement relationships are derived from general expressions for a thin shell of revolution. The strain and kinetic energies are determined in the co-rotating reference frame. The strain energy is first specified for large deformations and then split into a linear and a non-linear part. The nonlinear part, which is afterwards linearized, is necessary in order to take into account the effects of centrifugal and pressure pre-tensions. Both the Green-Lagrange nonlinear strains and the engineering strains are considered. The kinetic energy is formulated taking into account centrifugal and Coriolis terms. The variation of displacements $u$, $v$ and $w$ in the circumferential direction is described exactly. This is done by assuming appropriate trigonometric functions with a unique argument $n\varphi + \omega t$ in order to allow for rotating mode shapes. The dependence of the displacements on the meridional coordinate is described through Fourier series. The Rayleigh-Ritz method is applied to determine the Fourier coefficients. As a result, an ordinary stiffness matrix, a geometric stiffness matrix due to pressurisation and centrifugal forces, and three inertia matrices incorporating squares of natural frequencies, products of rotational speed and natural frequencies and squares of the rotational speed are derived. The application of the developed procedure is illustrated in the cases of a closed toroidal shell and a thin-walled toroidal ring. With the increase of the rotation speed the natural frequencies of most natural modes are split into two (bifurcate). The corresponding stationary modes are split into two modes rotating forwards and backwards around the circumference with different speeds. The obtained results are compared with FEM results and a very good agreement is observed. The advantage of the proposed semi-analytical method is high accuracy and low CPU time-consumption in case of small pre-stress
deformation for realistic structures. The illustrated numerical examples can be used as benchmark for validation of numerical methods.

**Keywords:** Toroidal shell, Vibration, Rotation, Centrifugal forces, Natural frequency bifurcation, Rayleigh-Ritz method, Fourier series

1. **Introduction**

A great deal of engineering structures have such geometry that they can be considered as shells. The mechanics of shells have been a subject of investigation for over a century. The main accomplishments are presented in number of books covering statics and/or dynamics of shells [1]-[6].

In case of complicated shell geometries, numerical methods are nowadays normally used. However, analytical or semi-analytical methods offer a more transparent interpretation of the results and can often serve as benchmarks for assessing the accuracy of numerical results. Analytical solutions can only be achieved for specific simplified geometries. These include for example beams, rings, plates, cylinders, spheres and tori [5]. Even with such simplified geometries, closed-form solutions are only possible and practical for certain combinations of boundary conditions.

In certain engineering situations an axisymmetric shell (shell of revolution) rotates around its axis of symmetry. This occurs, for example, with automotive tyres [7]-[11]. With rotating shell-like structures some interesting effects have been observed. These effects include shifts of natural frequencies due to centrifugal forces. This is because the centrifugal forces cause an initial “in-plane” membrane tension. In addition, the bifurcation of natural frequencies and rotating natural modes have been observed. For example, Bryan studied vibrations of a rotating ring and described the rotating mode phenomenon [12]. Later on, many researches contributed to the field by developing the methodology for studying vibrations of rotating rings (i.e. [13]-[15]), and cylinders, (i.e. [16]-[20]). Huang and Soedel’s paper on a simply supported rotating cylinder is instructive because it gives very clear mathematical and physical explanations for the phenomena of bifurcation of natural frequencies and rotating mode shapes [18].

Considering now vibrations of toroidal shells, the literature is considerably scarcer than that covering the vibration of, for example, plates or cylinders, as pointed out by Kang [21]. With toroidal shells, partial differential equations of motion can be reduced to a set of eight ordinary differential equations with variable coefficients. However, due to the variable
coefficients, it is very difficult to obtain straightforward closed-form analytical solutions. In this situation, Fourier series can be used to describe the displacements. Since trigonometric functions of increasing Fourier orders are linearly independent in the range 0-2π, it is possible to obtain an infinite number of sets of eight or less ordinary differential equations with constant coefficients [22]. It is interesting to mention that a similar methodology can be used to analyse the problem of elastic stability (buckling) of a closed toroidal shell [23],[24].

Energy approach is an alternative to directly solving the system of differential equation of motion. For example, Lincoln and Volterra considered free vibrations of toroidal rings theoretically and experimentally [25]. In the theoretical part of their study, they expressed the components of the elastic displacements of the ring in a Taylor's series expansion in terms of the radial and axial coordinates. The authors calculated the corresponding potential and kinetic energies of the ring and used the Hamilton's principle to determine the coefficients of the expansion. The theoretical results are also compared with results obtained experimentally on five steel toroids of different thicknesses [25].

Although in essence it is an approach based on the minimisation of energy and the variational integral, the Galerkin method has become a general algorithm for solving a variety of equations and problems [5]. For example, Leung and Kwok used the Galerkin method with complete Fourier series to describe the three displacement components of a toroidal shell segment (curved pipe), having a circular cross-section, as functions of the circumferential and meridional coordinates [26]. Ming et al. treated their curved pipe in a similar manner, however instead of using the Fourier series in both directions, they represented only the meridional mode profiles by trigonometric functions and for the circumferential mode profiles they used combinations of beam deflection functions which satisfy the boundary conditions [27]. Another application of Galerkin method has been used to investigate the effects of internal pressure on the natural frequencies of an inflated torus [28]. Some very recent works on vibration analysis of toroidal shells deal with tori made of composite layers [29], of variable thickness properties, [21], or include the effects of shear deformations and rotary inertia, [30], which are ordinarily neglected.

Considering now fully numerical procedures for the analysis of axisymmetric structures, an opened or a closed shell in the circumferential direction can be modelled by shell finite elements, [31], [32]. General formulation of doubly curved shell elements is presented in [33]. For vibration analysis of a shell closed in the circumferential direction special waveguide finite elements have been developed, [34]-[37]. In this case a 3D problem
is reduced to a 2D problem. Comparison of these two types of finite elements is presented in [38].

The present state-of-the art motivates to find a rigorous solution for the free vibrations problem of rotating and pressurised toroidal shells. The work in this paper is dedicated to this problem with a particular aim of better understanding the dynamic behaviour of rotating tires. For this purpose, the Rayleigh-Ritz method is used [39]. Ordinary strain energy, strain energy due to pre-stressing and the kinetic energy are formulated taking into account the variation of shell displacements in the circumferential direction exactly, by using simple trigonometric functions. Mode profiles of the shell cross-section (the variation in the meridional direction) are described by Fourier series. Minimizing the total energy by its differentiation per Fourier coefficients, a matrix equation of motion is obtained. The application of the presented numerical procedure is illustrated in the case of a closed toroidal shell and a thin-walled toroidal ring.

The organisation of the paper is as follows. In Section 2 general expressions of the ordinary strain and the strain energy due to the pre-stressing are formulated. Also the kinetic energy of an axisymmetric shell is derived in the co-rotating reference frame including both centrifugal and Coriolis terms. In Section 3 the analysis is narrowed down to the toroidal geometry. In Section 4 the stiffness and mass matrices are derived, and the eigenvalue problem is formulated. In Section 5 the application of the developed method is illustrated on two examples, a toroidal shell with ordinary dimensions and a thin toroidal ring. The paper also contains six appendices. In appendices A-D variable coefficients of linear and non-linear strain energies, and submatrices of the stiffness and mass matrices are specified. Appendix E discusses lower order strain and kinetic energy terms, and Appendix F deals with the determination of tension forces due to the centrifugal load.

2. General strain – displacement relationships and energy expressions

2.1. Strain energy

Love's simplification introduced in the thin shell theory [1] enables to decouple a thin shell strain field into membrane strains due to extensional deformations and bending strains due to curvature changes

\[ \tilde{\varepsilon}_{11} = \varepsilon_{11} + z\kappa_{11}, \quad \tilde{\varepsilon}_{22} = \varepsilon_{22} + z\kappa_{22}, \quad \tilde{\varepsilon}_{12} = \varepsilon_{12} + z\kappa_{12}, \]

where \( z \) is the distance of a shell layer from the reference mid-surface. General expressions for the membrane strains and bending strains are, respectively [5].
\[ \varepsilon_{11} = \frac{1}{A_1} \frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} + \frac{u_3}{R_1} \]
\[ \varepsilon_{22} = \frac{1}{A_2} \frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} + \frac{u_3}{R_2} \]
\[ \varepsilon_{12} = \frac{A_2}{A_1} \frac{\partial u_2}{\partial \alpha_1} + \frac{A_1}{A_2} \frac{\partial u_1}{\partial \alpha_2} \]

(2)

\[ \kappa_{11} = \frac{1}{A_1} \frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1} \frac{\partial A_1}{\partial \alpha_2} \]
\[ \kappa_{22} = \frac{1}{A_2} \frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_2} \frac{\partial A_2}{\partial \alpha_1} \]
\[ \kappa_{12} = \frac{A_2}{A_1} \frac{\partial \beta_2}{\partial \alpha_1} + \frac{A_1}{A_2} \frac{\partial \beta_1}{\partial \alpha_2} \]

(3)

where

\[ \beta_1 = \frac{u_2}{R_1} - \frac{1}{A_1} \frac{\partial u_3}{\partial \alpha_1} \]
\[ \beta_2 = \frac{u_2}{R_2} - \frac{1}{A_2} \frac{\partial u_3}{\partial \alpha_2} \]

(4)

are rotation angles. The shell geometric parameters are defined in the curvilinear surface coordinate system by coordinates \( \alpha_1 \) and \( \alpha_2 \). Symbols \( A_1 \) and \( A_2 \) represent the two Lamé parameters, whereas \( R_1 \) and \( R_2 \) are the two radii of curvatures. \( u_1 \) and \( u_2 \) are the extensional, “in-plane”, displacements and \( u_3 \) is the “out of plane” deflection.

Stresses in a shell layer are defined according to the two-dimensional Hooke's law

\[ \sigma_{11} = \frac{E}{1-\nu^2} (\varepsilon_{11} + \nu \varepsilon_{22}), \quad \sigma_{22} = \frac{E}{1-\nu^2} (\varepsilon_{22} + \nu \varepsilon_{11}), \quad \sigma_{12} = \frac{E}{2(1+\nu)} \varepsilon_{12}, \]

(5)

where \( E \) is Young's modulus and \( \nu \) is Poisson's ratio.

The strain energy stored in a shell of thickness \( h \) is

\[ E_s = \frac{1}{2} \int_{A} \int_{-h/2}^{h/2} \left( \sigma_{11} \varepsilon_{11} + \sigma_{22} \varepsilon_{22} + \sigma_{12} \varepsilon_{12} \right) dz \, dA, \]

(6)

where \( dA = A_1 A_2 \, d\alpha_1 d\alpha_2 \) is an infinitesimal mid-surface area. Substituting (5) into (6) and integrating (6) over the shell thickness, yields
\[
E_i = \frac{1}{2} \int \int \left\{ \frac{\alpha_i^2 + \beta_i^2 + 2\nu \alpha_{i1} \beta_{i2} + \frac{1}{2} (1 - \nu) \beta_{i2}^2}{\int \int \left[ \frac{K}{A} \left[ \alpha_{i1}^2 + \alpha_{i2}^2 + 2\nu \alpha_{i1} \alpha_{i2} + \frac{1}{2} (1 - \nu) \beta_{i2}^2 \right] + D \left[ \kappa_{i1}^2 + \kappa_{i2}^2 + 2\nu \kappa_{i1} \kappa_{i2} + \frac{1}{2} (1 - \nu) \kappa_{i2}^2 \right] \right] A_1 A_2 \, d \alpha_1 \, d \alpha_2 \right\},
\]

where

\[
K = \frac{Eh}{1 - \nu^2}, \quad D = \frac{Eh^3}{12(1 - \nu^2)}
\]

are the membrane and bending stiffnesses, respectively. Note that the membrane and bending strain energies are uncoupled in (7). This is because functions \( \epsilon_{ij} \) and \( \kappa_{ij} \) are orthogonal within the shell thickness domain.

By substituting Eqs. (2) and (3) into Eq. (7), the shell strain energy can be expressed in terms of the three displacement components.

### 2.2. Strain energy due to pre-stressing

It is well-known from the plate stability theory that membrane forces can cause buckling. The membrane forces are imposed on the deformed plate due to bending, and the problem is considered to be a nonlinear one with large displacements. Hence, for the buckling analysis, the second order strains are normally taken into account, [40]. Similarly, in order to analyse vibrations of tensioned plates, it is also necessary to consider the second order strains. Membrane forces produced by the pre-tension can cause a considerable shift of the plate natural frequencies in comparison to the untensioned plate.

In case of non-flat shell-like structures, such as cylinders, spheres or tori, the problem becomes more complicated. If such a structure has an axis of symmetry and rotates around it, like for example automotive tyres, then the membrane forces are caused either by the internal pressure or by a combination of the internal pressure and a centrifugal tension.

The second order strains in a thin shell structure can be determined based on the expansion of the Green-Lagrange tensor into the Cartesian coordinate system [41]. In order to avoid complicated transformation of the orthogonal coordinates to curvilinear ones, the analogy of physical meaning of different expanded terms can be used. Hence, one can write for the second order strains
\[ \varepsilon_{11}^* = \frac{1}{2} \left[ \varepsilon_{11}^2 + \left( \varepsilon_{12}^{(1)} \right)^2 + \beta_1^2 \right], \]
\[ \varepsilon_{22}^* = \frac{1}{2} \left[ \varepsilon_{22}^2 + \left( \varepsilon_{12}^{(2)} \right)^2 + \beta_2^2 \right]. \]

(9)

where \( \varepsilon_{11} \) and \( \varepsilon_{22} \) are tensional strains, \( \varepsilon_{12}^{(1)} \) and \( \varepsilon_{12}^{(2)} \) are parts of the shear strain \( \varepsilon_{12} \) with displacement variation in \( \alpha_1 \) and \( \alpha_2 \) direction respectively, Eqs. (2), and \( \beta_1 \) and \( \beta_2 \) are the rotation angles, Eqs. (4).

Furthermore, besides the first and the second order strains, there are also additional strains due to pre-stressing displacements. Hence, the total displacements are \( u_i + u_i^0 \), \( u_2 + u_2^0 \), and \( u_3 + u_3^0 \). If those expressions are substituted into Eqs. (2) and then the result of substitution into (9), and if small terms of higher order are omitted, one obtains formulae for the total strains

\[ \varepsilon_{11}^{(i)} = \varepsilon_{11} + \varepsilon_{11}^0 + \varepsilon_{11}^0, \quad \varepsilon_{22}^{(i)} = \varepsilon_{22} + \varepsilon_{22}^0 + \varepsilon_{22}^0, \quad \varepsilon_{12}^{(i)} = \varepsilon_{12}. \]

(10)

where \( \varepsilon_{11} \), \( \varepsilon_{22} \) and \( \varepsilon_{12} \) are given with Eqs. (2) and \( \varepsilon_{11}^* \) and \( \varepsilon_{22}^* \) with Eqs. (9). In the considered case, pre-stressing is a result of action of the membrane forces

\[ \varepsilon_{11}^0 = \frac{1}{Eh} (N_1 - \nu N_2), \quad \varepsilon_{22}^0 = \frac{1}{Eh} (N_2 - \nu N_1). \]

(11)

Hence, the total strains consist of three terms.

The total tensional strain energy is presented in the form of the first integral in Eq. (7), i.e.

\[ E_{\text{S}}^{(i)} = \frac{1}{2} \int \int K \left[ \left( \varepsilon_{11}^{(i)} \right)^2 + \left( \varepsilon_{22}^{(i)} \right)^2 + 2\nu \varepsilon_{11}^{(i)} \varepsilon_{22}^{(i)} + \frac{1}{2} (1 - \nu) \varepsilon_{12}^{(i)} \right] A_1 A_2 \, d\alpha_1 \, d\alpha_2. \]

(12)

Substituting Eq. (11) into (12) and expanding it, one can observe, based on Eqs. (2) and (9), terms with different order of displacements: \( \delta^0 \), \( \delta^1 \), \( \delta^2 \), \( \delta^3 \), \( \delta^4 \). In the present study of linear natural vibrations by the Rayleigh-Ritz method, only the second order terms are relevant for formulation of the eigenvalue problem. The physical meaning of the \( \delta^0 \) and \( \delta^1 \) terms is analysed later on. The \( \delta^3 \) and \( \delta^4 \) terms are negligible small quantities of higher order. Hence, one can write \( E_{\text{S}}^{(i)} = E_{\text{St}} + E_G \), where \( E_{\text{St}} \) is the linear tensional strain energy contained in the first term of Eq. (7), and

\[ E_G = \int \int K \left[ \varepsilon_{11}^0 \left( \varepsilon_{11}^0 + \nu \varepsilon_{22}^0 \right) + \varepsilon_{12}^0 \left( \varepsilon_{22}^0 + \nu \varepsilon_{12}^0 \right) \right] A_1 A_2 \, d\alpha_1 \, d\alpha_2. \]

(13)
is the strain energy due to pre-stressing. Since according to (11)

\[ \varepsilon_1^0 + v_2^0 = \frac{N_1}{K}, \quad \varepsilon_2^0 + v_1^0 = \frac{N_2}{K} \]  

Eq. (13) can be reduced to

\[ E_G = \int \int (\varepsilon_1^* N_1 + \varepsilon_2^* N_2) A_i A_j d\alpha_i d\alpha_j, \]  

where \( \varepsilon_1^* \) and \( \varepsilon_2^* \) are given by Eqs. (9). Keeping only the dominant terms in Eqs. (9) yields

\[ \varepsilon_{11}^* = \frac{1}{2A_1^2} \left[ \left( \frac{\partial u_1}{\partial \alpha_1} \right)^2 + \left( \frac{\partial u_2}{\partial \alpha_1} - \frac{u_2}{A_1} \frac{\partial A_2}{\partial \alpha_1} \right)^2 + \left( \frac{\partial u_3}{\partial \alpha_1} \right)^2 \right] \]  

\[ \varepsilon_{22}^* = \frac{1}{2A_2^2} \left[ \left( \frac{\partial u_1}{\partial \alpha_2} \right)^2 + \left( \frac{\partial u_2}{\partial \alpha_2} \right)^2 + \left( \frac{\partial u_3}{\partial \alpha_2} \right)^2 \right]. \]  

An approximate formulation for the second order strains, derived in [5] in a more intricate manner, neglecting small quantities of higher order, reads

\[ \varepsilon_{11}^* = \frac{1}{2A_1^2} \left[ \left( \frac{\partial u_1}{\partial \alpha_1} - \frac{u_1}{A_1} \frac{\partial A_1}{\partial \alpha_1} \right)^2 + \left( \frac{\partial u_2}{\partial \alpha_1} - \frac{u_2}{A_1} \frac{\partial A_2}{\partial \alpha_1} \right)^2 + \left( \frac{\partial u_3}{\partial \alpha_1} \right)^2 \right] \]  

\[ \varepsilon_{22}^* = \frac{1}{2A_2^2} \left[ \left( \frac{\partial u_1}{\partial \alpha_2} - \frac{u_1}{A_1} \frac{\partial A_1}{\partial \alpha_2} \right)^2 + \left( \frac{\partial u_2}{\partial \alpha_2} \right)^2 + \left( \frac{\partial u_3}{\partial \alpha_2} \right)^2 \right]. \]  

By comparing Eqs. (17) with (16) it is observed that Eqs. (17) contains some additional terms.

A new formulation of the nonlinear strain-displacement relation is presented in [42] for large local displacements and rotation but small strain vibrations. The local rigid-body displacements from the total displacements are removed and small local displacement field is used to derive objective local engineering strains. This is done by omitting the first terms in Eqs. (9).

2.3. Kinetic energy of a rotating shell of revolution

A shell of revolution with the particle \( P(\theta, \phi) \) as the origin of moving coordinate system defined by the unit vectors \( \vec{e}_1, \vec{e}_2 \) and \( \vec{e}_3 \), and the corresponding displacements \( u_1, u_2 \) and \( u_3 \), is shown in Fig. 1. The particle velocity vector consists of three parts, i.e. due to shell rotation with speed \( \Omega \), relative shell vibration and oscillation in the rotating field, [5]

\[ \vec{v}_p = \vec{v}_0 + \vec{v}_\theta + \vec{v}_\phi, \]  

in which
\[ \ddot{v}_\Omega = r\Omega \dot{e}_2 \]
\[ \ddot{v}_\delta = \frac{\partial \ddot{\delta}}{\partial t} = \dot{\delta} \]
\[ \dddot{v}_{\Omega \delta} = \ddot{\Omega} \times \dot{\delta} \]
\[ \dddot{\Omega} = \Omega \cos \theta \ddot{e}_3 - \Omega \sin \theta \dot{e}_1 \]
\[ \dddot{\delta} = u_1 \ddot{e}_1 + u_2 \ddot{e}_2 + u_3 \ddot{e}_3 \]
\[ \dddot{\delta} = \dot{u}_1 \dot{e}_1 + \dot{u}_2 \dot{e}_2 + \dot{u}_3 \dot{e}_3, \]

where \( t \) is time. Substituting Eqs. (19) into (18) yields
\[ \ddot{v}_p = (\dot{u}_1 - u_2 \Omega \cos \theta) \ddot{e}_1 + (\Omega \dot{r} + \dot{u}_2 + u_4 \Omega \cos \theta + u_5 \Omega \sin \theta) \dot{e}_2 + (\dot{u}_3 - u_2 \Omega \sin \theta) \dot{e}_3. \]

The kinetic energy is
\[ E_k = \frac{1}{2} \rho h \int \int_{\alpha_1, \alpha_2} \dot{v}_p \cdot \dot{v}_p \, A_1 A_2 \, d\alpha_1 \, d\alpha_2, \]
and substituting Eq. (20) into (21), one obtains
\[ \begin{aligned}
&= \frac{1}{2} \rho h \int \int_{\alpha_1, \alpha_2} \left[ \left( \frac{\partial u_1}{\partial t} - u_2 \Omega \cos \theta \right)^2 + \left( \Omega \dot{r} + \frac{\partial u_2}{\partial t} + u_4 \Omega \cos \theta + u_5 \Omega \sin \theta \right)^2 \\
&+ \left( \frac{\partial u_3}{\partial t} - u_2 \Omega \sin \theta \right)^2 \right] A_1 A_2 \, d\alpha_1 \, d\alpha_2.
\end{aligned} \]

3. Toroidal shell
3.1. Strain-displacement relationships and strain energy

A toroidal shell with the main dimensions and displacements is shown in Fig. 2. The shell parameters are the following:

\[ A_1 = a, \quad A_2 = r, \quad \alpha_1 = \theta, \quad \alpha_2 = \varphi, \]
\[ r = R + a \sin \theta, \quad R_1 = a, \quad R_2 = \frac{r}{\sin \theta}, \]
\[ u_1 = u, \quad u_2 = v, \quad u_3 = w. \]

The in-plane strains, bending curvatures and rotation angles, Eqs. (2), (3) and (4) respectively, take the following form:
\[
\begin{align*}
\varepsilon_\vartheta &= \frac{1}{a} \left( \frac{\partial u}{\partial \vartheta} + w \right) \\
\varepsilon_\varphi &= \frac{1}{r} \left( \frac{\partial v}{\partial \varphi} + u \cos \varphi + w \sin \varphi \right) \\
\varepsilon_{\varphi \vartheta} &= \frac{1}{r \frac{\partial \varphi}{\partial \varphi}} \left( \frac{\partial u}{a \frac{\partial \vartheta}{\partial \varphi}} + r \frac{\partial v}{\partial \varphi} - \frac{\cos \varphi}{r} \right),
\end{align*}
\]

(24)

\[
\begin{align*}
\kappa_\vartheta &= \frac{1}{a^2} \left( \frac{\partial u}{\partial \vartheta} - \frac{\partial^2 w}{\partial \vartheta^2} \right) \\
\kappa_\varphi &= \frac{\cos \vartheta}{ar} \left( u - \frac{\partial w}{\partial \vartheta} \right) + \frac{1}{r^2} \left( \frac{\partial v}{\partial \varphi} \sin \vartheta - \frac{\partial^2 w}{\partial \varphi^2} \right) \\
\kappa_{\varphi \vartheta} &= \frac{1}{ar} \left[ \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial \vartheta} \sin \vartheta + \cos \vartheta \left( 1 - 2 \frac{a}{r} \sin \vartheta \right) v - 2 \frac{\partial w}{\partial \vartheta} + 2 \frac{a}{r} \cos \vartheta \frac{\partial w}{\partial \varphi} \right],
\end{align*}
\]

(25)

\[
\begin{align*}
\beta_1 &= \frac{1}{a} \left( u - \frac{\partial w}{\partial \vartheta} \right) \\
\beta_2 &= \frac{1}{r} \left( v \sin \vartheta - \frac{\partial w}{\partial \varphi} \right),
\end{align*}
\]

(26)

For a toroidal shell closed in the circumferential direction, with an either open or a closed cross-section, having arbitrary cross-sectional boundary conditions, the displacement components can be assumed in the form

\[
\begin{align*}
u(\vartheta, \varphi, t) &= U(\vartheta) \cos(n \varphi + \omega t) \\
v(\vartheta, \varphi, t) &= V(\vartheta) \sin(n \varphi + \omega t) \\
w(\vartheta, \varphi, t) &= W(\vartheta) \cos(n \varphi + \omega t),
\end{align*}
\]

(27)

where functions \( U(\vartheta) \), \( V(\vartheta) \) and \( W(\vartheta) \) are the meridional, circumferential and radial displacement components of the cross-section, respectively, and \( \omega \) is the natural frequency.

Substituting (27) into Eqs. (24) and (25), and then into the strain energy (7), one obtains products of two displacement amplitudes or their derivatives, with squares of sine and cosine functions (27). Their integral over the circumferential angle \( \varphi \) within the domain \( 0 - 2\pi \) equals \( \pi \). Thus the temporal variation vanishes and the strain energy becomes time-invariant. This is due to the fact that the modes rotate while keeping a fixed circumferential profile. The integral over the meridional coordinate \( \vartheta \) is for the moment left open and it reads
\[ E_i = \int \left[ \frac{1}{2} p_1 (U')^2 + \frac{1}{2} p_2 U'^2 + p_3 U' + \frac{1}{2} p_4 (V')^2 + \frac{1}{2} p_3 V'^2 + p_6 V \right. \\
+ p_7 U' + p_8 UV' + p_9 UV \\
+ \frac{1}{2} q_1 (W')^2 + \frac{1}{2} q_2 (W')^2 + \frac{1}{2} q_3 W'^2 + q_4 W' W + q_5 W W + q_6 WW \\
+ q_7 W' U' + q_8 (W' U + W U') + q_9 W' U + q_{10} W U + q_{11} W U \\
+ q_{12} W V' + q_{13} W V' + q_{14} W V + q_{15} W V' + q_{16} W V \left] d \vartheta, \right. \]

where \( p_i (\vartheta), i = 1, 2 \ldots 9 \) and \( q_j (\vartheta), j = 1, 2 \ldots 16 \) are variable coefficients specified in Appendix A.

### 3.2. Strain energy due to pre-stressing

Taking into account the geometry of a toroidal shell, Eq. (23), the in-plane strains, Eq. (24), and rotation angles, Eq. (26), the general expression for the strain energy component due to initial membrane forces, Eq. (16), takes the form

\[ E_G = \frac{1}{2} \int_0^{2\pi} \int_0^a \left[ \frac{\partial u}{\partial \vartheta} + w \right]^2 + \left( \frac{\partial v}{\partial \vartheta} - \frac{a}{r} v \cos \vartheta \right)^2 + \left( u - \frac{\partial w}{\partial \vartheta} \right)^2 \right] N_{\vartheta} \\
+ \frac{a}{r} \left[ \frac{\partial u}{\partial \varphi} \right]^2 + \left( \frac{\partial v}{\partial \varphi} + u \cos \vartheta + w \sin \vartheta \right)^2 + \left( v \sin \vartheta - \frac{\partial w}{\partial \varphi} \right)^2 \right] N_{\varphi} \, d \vartheta d \varphi. \]  

Substituting Eqs. (27) into (29), and integrating over the circumferential coordinate \( \varphi \), one obtains again a time-invariant expression

\[ E_G = \int \left[ \frac{1}{2} c_1 (U')^2 + \frac{1}{2} c_2 U'^2 + \frac{1}{2} c_3 (V')^2 + \frac{1}{2} c_4 V'^2 + c_5 V \right. \\
+ \frac{1}{2} c_6 (W')^2 + \frac{1}{2} c_7 W'^2 + c_8 (W' U + W U') + c_{10} W U + c_{11} W U \left] d \vartheta, \right. \]

where the variable coefficients \( c_i (\vartheta), i = 1, 2 \ldots 11 \) are given in Appendix B.

### 3.3. Kinetic energy of rotating shell

The rotating toroidal shell vibrates with respect to the statically deformed geometry due to pre-stressing. As a result, total displacements consist of time variable and constant part

\[ u_i (t) = u(t) + u_0, \quad u_2 (t) = v(t) + v_0, \quad u_3 (t) = w(t) + w_0. \]
If the toroidal shell parameters (23) and Eqs. (31) are taken into account, it is obvious that the expanded Eq. (22) for kinetic energy contains three sets of terms, which depend on different order of vibration displacements: \( \delta^0, \delta^1, \delta^2 \). Only \( \delta^2 \) terms are relevant for natural vibration analysis. The physical meaning of the \( \delta^0 \) and \( \delta^1 \) terms is analysed later on.

According to the above consideration, the kinetic energy of shell vibrations reads

\[
E_k = \frac{1}{2} \rho h \int_0^{2\pi} \left[ \left( \frac{\partial u}{\partial t} - \nu \Omega \cos \vartheta \right)^2 + \left( \frac{\partial v}{\partial t} + \mu \Omega \sin \vartheta + w \Omega \sin \vartheta \right)^2 \right. \\
+ \left. \left( \frac{\partial w}{\partial t} - \nu \Omega \sin \vartheta \right)^2 \right] ar d\vartheta d\phi.
\]

Substituting Eqs. (27) into (32), and integrating over \( \varphi \), one obtains

\[
E_k = \frac{1}{2} \pi \rho ha \int_0^{2\pi} \left[ \left( \omega^2 + \Omega^2 \cos^2 \vartheta \right) U^2 + \left( \omega^2 + \Omega^2 \right) V^2 + \left( \omega^2 + \Omega^2 \sin^2 \vartheta \right) W^2 \right. \\
+ \left. 4\omega \Omega \left( \cos \vartheta UV + \sin \vartheta VW \right) + 2\Omega^2 \sin \vartheta \cos \vartheta UW \right] d\vartheta.
\]

The kinetic energy is also time-invariant as the strain energies.

### 3.4. Membrane forces due to internal pressure and centrifugal load

An infinitesimal element of a rotating shell of revolution, with generalised dimensions, internal pressure, \( p \), centrifugal load, \( q \), and membrane forces \( N_1 \) and \( N_2 \), is shown in Fig. 3. According to the membrane theory of shells of revolution, [6], the equilibrium of forces in the meridional (tangential) and radial (normal) directions yields

\[
\frac{1}{r R_1} \left[ \frac{d(r N_1)}{d \vartheta} - \frac{d r}{d \vartheta} N_2 \right] = -q_1
\]

\[
\frac{N_1}{R_1} + \frac{N_2}{R_2} = p + q_a,
\]

where

\[
q_1 = \rho h \Omega^2 r \cos \vartheta, \quad q_a = \rho h \Omega^2 r \sin \vartheta
\]

are the two components of the centrifugal load \( q = \rho h \Omega^2 r \).

For a toroidal shell \( R_1 = a \) and \( R_2 = R/\sin \vartheta \). The differential equations (34) are solved separately for the case of pressure load \( p \) and centrifugal load \( q \) in two different ways in order to avoid a singular (trivial) solution at \( \vartheta = 0 \). In the first case force \( N_1 \) is eliminated from the system (34) and the resulting equation reads
\[ \frac{dN_2}{d\theta} \sin \theta + 2N_2 \cos \theta = pa \cos \theta. \] (36)

Force \( N_2 \) is obtained from (36), as a constant quantity, and \( N_1 \) from the second of Eqs. (34) as,

\[ N_2 = \frac{1}{2} pa, \quad N_1 = \frac{1}{2} pa \frac{R+r}{r}. \] (37)

In the second case, force \( N_2 \) is eliminated from (34) and one finds

\[ \frac{d}{d\theta} (r \sin \theta N_1) = ar(q_a \cos \theta - q_1 \sin \theta). \] (38)

Taking into account Eqs. (35), the right hand side of Eq. (38) equals zero and one finds \( N_1 = Cl(r \sin \theta) \), i.e. \( N_1 = 0 \) due to singularity at \( \theta = 0 \). Force \( N_2 \) is obtained from the second of Eqs. (34)

\[ N_2 = \rho h \Omega^2 r^2. \] (39)

Hence, the total membrane forces read

\[ N_\theta = N_{1\rho} + N_{1\phi} = \frac{1}{2} pa \frac{R+r}{r} \]
\[ N_\phi = N_{2\rho} + N_{2\phi} = \frac{1}{2} pa + \rho h \Omega^2 r^2. \] (40)

If the ratio \( a/R \) approaches zero, then the force \( N_{1\rho} \) converges to \( pa \), Eq. (40). Values \( N_{1\rho} = pa \) and \( N_{2\rho} = pa/2 \) are actually related to a thin-walled ring and correspond to a cylinder circumferential and axial force, \( N_{\phi \rho} = pa \) and \( N_{\phi \phi} = pa/2 \), respectively.

4. Application of the Rayleigh-Ritz method

4.1. Displacement field and stiffness matrix

If a closed toroidal shell is considered, then there are no discontinuities or boundary conditions. Therefore, shell displacements can be assumed in the form of a complete Fourier series

\[ U(\theta) = \sum_{m=0}^\infty A_m \cos m \theta + \sum_{m=0}^\infty B_m \sin m \theta \]
\[ V(\theta) = \sum_{m=0}^\infty C_m \cos m \theta + \sum_{m=0}^\infty D_m \sin m \theta \] (41)
\[ W(\theta) = \sum_{m=0}^\infty E_m \cos m \theta + \sum_{m=0}^\infty F_m \sin m \theta, \]
where $A_m$, $B_m$, $C_m$, $D_m$, $E_m$ and $F_m$ are the unknown Fourier coefficients. In spite of the fact that $B_0 = D_0 = F_0 = 0$, these coefficients are kept in order to generalize the procedure.

Displacements (41) can be presented in a matrix notation

$$
U(\vartheta) = \begin{bmatrix}
\langle f_m \rangle \\
\langle g_m \rangle
\end{bmatrix}
\begin{bmatrix}
\{A_m\} \\
\{B_m\}
\end{bmatrix},
$$

$$
V(\vartheta) = \begin{bmatrix}
\langle f_m \rangle \\
\langle g_m \rangle
\end{bmatrix}
\begin{bmatrix}
\{C_m\} \\
\{D_m\}
\end{bmatrix},
$$

$$
W(\vartheta) = \begin{bmatrix}
\langle f_m \rangle \\
\langle g_m \rangle
\end{bmatrix}
\begin{bmatrix}
\{E_m\} \\
\{F_m\}
\end{bmatrix},
$$

(42)

where

$$f_m = \cos m\vartheta, \quad g_m = \sin m\vartheta, \quad m = 0, 1, 2...N. \quad (43)$$

Substituting expressions (42) into (28) and differentiating the strain energy per Fourier coefficients, a system of three matrix equations is obtained
\[
\begin{align*}
\frac{\partial E_{x}}{\partial A_{i}} &= \int_{0}^{2\pi} \left( p_{1}^{k}_{1} + p_{2}^{k}_{2} + p_{3}^{k}_{3} \right) d\omega \left\{ A_{m} \right\} \\
\frac{\partial E_{x}}{\partial B_{i}} &= \int_{0}^{2\pi} \left( p_{1}^{k}_{1} + p_{2}^{k}_{2} + p_{3}^{k}_{3} \right) d\omega \left\{ B_{m} \right\} \\
\frac{\partial E_{x}}{\partial C_{i}} &= \int_{0}^{2\pi} \left( q_{1}^{k}_{1} + q_{2}^{k}_{2} + q_{3}^{k}_{3} \right) d\omega \left\{ C_{m} \right\} \\
\frac{\partial E_{x}}{\partial D_{i}} &= \int_{0}^{2\pi} \left( q_{1}^{k}_{1} + q_{2}^{k}_{2} + q_{3}^{k}_{3} \right) d\omega \left\{ D_{m} \right\} \\
\frac{\partial E_{x}}{\partial E_{i}} &= \int_{0}^{2\pi} \left( q_{1}^{k}_{1} + q_{2}^{k}_{2} + q_{3}^{k}_{3} \right) d\omega \left\{ E_{m} \right\} \\
\frac{\partial E_{x}}{\partial F_{i}} &= \int_{0}^{2\pi} \left( q_{1}^{k}_{1} + q_{2}^{k}_{2} + q_{3}^{k}_{3} \right) d\omega \left\{ F_{m} \right\}
\end{align*}
\]

where \( p_{i}(\omega), i = 1, 2, \ldots, 9 \) and \( q_{i}(\omega), i = 1, 2, \ldots, 16 \) are variable coefficients depending on the meridional coordinate \( \omega \) which are specified in Appendix A. Submatrices \( [k] \), whose elements are products of sine and cosine functions or their derivatives per \( \omega \), are listed in Appendix C.

The system of three matrix equations (44) can be presented in a condensed form

\[
\begin{align*}
\frac{\partial E_{x}}{\partial [\omega]} &= [K][\omega],
\end{align*}
\]

where,
\[
\{\delta\}^T = \{\delta\} = \{\langle A_m \rangle \langle B_m \rangle \langle C_m \rangle \langle D_m \rangle \langle E_m \rangle \langle F_m \rangle\}
\]

is the vector of Fourier coefficients, and

\[
[K] = \begin{bmatrix} [K]_{11} & [K]_{12} & [K]_{13} \\ [K]_{21} & [K]_{22} & [K]_{23} \\ [K]_{31} & [K]_{32} & [K]_{33} \end{bmatrix},
\]

is the stiffness matrix. Submatrices \([K]_{ij}, i, j = 1, 2, 3\), encompass the integrals in Eq. (44).

4.2. Geometric stiffness matrix

Geometric stiffness matrix is derived from the strain energy component which is due to pre-stressing, Eq. (30). Substituting expressions (42) for displacements into (30) and differentiating it per Fourier coefficients, the following three matrix equations are obtained,

\[
\frac{\partial E_G}{\partial A_k} = \int_0^{2\pi} (c_1 [k]_1 + c_2 [k]_2) d\vartheta \{A_m\} + \int_0^{2\pi} c_8 [k]_2 d\vartheta \{C_m\},
\]

\[
\frac{\partial E_G}{\partial B_k} = \int_0^{2\pi} c_8 [k]_2 d\vartheta \{B_m\} + \int_0^{2\pi} c_9 \left([k]_2^0 - [k]_2^1\right) + c_{10} [k]_2 d\vartheta \{F_m\},
\]

\[
\frac{\partial E_G}{\partial C_k} = \int_0^{2\pi} c_8 [k]_2 d\vartheta \{B_m\} + \int_0^{2\pi} c_9 \left(c_1 [k]_1 + c_4 [k]_2 + c_5 \left([k]_2^0 + [k]_2^1\right)\right) d\vartheta \{C_m\} + \int_0^{2\pi} c_{11} [k]_2 d\vartheta \{F_m\},
\]

\[
\frac{\partial E_G}{\partial D_k} = \int_0^{2\pi} c_9 \left([k]_2^0 - [k]_2^1\right) + c_{10} [k]_2 d\vartheta \{C_m\} + \int_0^{2\pi} c_{11} [k]_2 d\vartheta \{F_m\},
\]

\[
\frac{\partial E_G}{\partial E_k} = \int_0^{2\pi} \left(c_6 \left([k]_1^0 - [k]_1^1\right) + c_7 [k]_2\right) d\vartheta \{E_m\},
\]

where variable coefficients \(c_i(\vartheta), i = 1, 2...11\) for the Green-Lagrange nonlinear strain and the engineering strain are specified in Appendix B. The submatrices \([k]\) are given in Appendix C.
The system of three matrix equations (48) can be presented in a condensed form by following the layout of Eq. (45)

$$\frac{\partial E_G}{\partial \{\delta\}} = [G]\{\delta\}. \quad (49)$$

where

$$[G] = \begin{bmatrix} [G]_{11} & [G]_{12} & [G]_{13} \\ [G]_{21} & [G]_{22} & [G]_{23} \\ [G]_{31} & [G]_{32} & [G]_{33} \end{bmatrix} \quad (50)$$

is the geometric stiffness matrix. Submatrices $[G]_{ij}, i, j = 1, 2, 3,$ now represent the integrals in Eqs. (48).

According to the composition of the membrane forces $N_\theta$ and $N_\phi$, Eqs. (40), the geometric stiffness matrix can be split into two matrices, i.e. the one which is due to the internal pressure and the other one related to the centrifugal forces, i.e.

$$[G] = p[G]_p + \Omega^2[G]_\Omega. \quad (51)$$

### 4.3. Mass matrices

Mass matrices are derived from the kinetic energy, Eq. (33). By substituting expressions (42) into (33), and differentiating the kinetic energy per Fourier coefficients, one obtains the following system of algebraic equations
\[
\begin{aligned}
\frac{\partial E_k}{\partial A_k} &= \alpha \left( \omega^2 + \Omega^2 \sin^2 \theta \right) \int_0^{2\pi} r \, d\theta \left\{ A_m \right\}, \\
\frac{\partial E_k}{\partial B_k} &= \alpha \left( \omega^2 + \Omega^2 \sin^2 \theta \right) \int_0^{2\pi} r \, d\theta \left\{ B_m \right\}, \\
\frac{\partial E_k}{\partial C_k} &= 2\alpha \omega \Omega \left( \omega^2 + \Omega^2 \right) \int_0^{2\pi} r \, d\theta \left\{ C_m \right\}, \\
\frac{\partial E_k}{\partial D_k} &= 2\alpha \omega \Omega \left( \omega^2 + \Omega^2 \right) \int_0^{2\pi} r \, d\theta \left\{ D_m \right\}, \\
\frac{\partial E_k}{\partial E_k} &= \alpha \Omega^2 \left( \omega^2 + \Omega^2 \sin^2 \theta \right) \int_0^{2\pi} r \, d\theta \left\{ E_m \right\}, \\
\frac{\partial E_k}{\partial F_k} &= 2\alpha \omega \Omega \left( \omega^2 + \Omega^2 \sin^2 \theta \right) \int_0^{2\pi} r \, d\theta \left\{ F_m \right\},
\end{aligned}
\]

where \( \alpha = \pi \rho_h \alpha \). The three matrix equations (52) can be presented in the form

\[
\frac{\partial E_k}{\partial \{ \theta \}} = \left( \Omega^2 [B] + \omega \Omega [C] + \omega^2 [M] \right) \left\{ \theta \right\},
\]

where
are mass matrices related to the centrifugal force \((\Omega^2)\), Coriolis force \((\omega \Omega)\), and the ordinary inertia force \((\omega^2)\). Submatrices \([B]_i, [C]_i, [M]_i\), \(i, j = 1, 2, 3\) are specified in Appendix D. They all depend on the symmetric matrix \([k]\), Appendix B. Therefore, all mass matrices (54), including the Coriolis matrix, are symmetric.

It is interesting to point out that the Coriolis (gyroscopic) matrix \([C]\) is antisymmetric (skew-symmetric) if the vibration problem of rotating structures is solved by differential equations of motion, [43]. The antisymmetric matrix \([C]\) is also obtained in the FEM formulation of the eigenvalue problem, [44]. If the Rayleigh-Ritz method with orthogonal coordinate functions or the finite strip method is applied, as for instance in the case of rotating cylindrical shell, [19],[45], a symmetric Coriolis matrix is obtained as in this paper.

### 4.4. Matrix equation of motion

If a linear conservative dynamic system vibrates at its natural frequency, then it interchanges vibration energy from a purely potential state with the maximum strain energy, \(E_{s_{\text{max}}}\), to a purely kinetic state where the kinetic energy is maximum \(E_{k_{\text{max}}}\), [5]. Hence, the difference of the maximum energies, \(\Pi = E_{s_{\text{max}}} - E_{k_{\text{max}}}\), equals zero. If these energies are determined for approximated mode shapes, then the difference \(\Pi\) is not zero. However, for a successful approximation of the true mode shape it should be as close to zero as possible.

In the considered case of a rotating toroidal shell the balance of energies reads, [19]

\[
\Pi = E_{s} + E_{G} - E_{k}.
\]

Here, the situation is somewhat different, since all the terms on the right hand side are time-invariant. This time-invariance is only due to the fact that fixed mode profiles rotate around the axis of symmetry of the torus. A natural frequency is in fact the speed of this rotation (see...
Eq. (27)). Then the integration over the circumferential coordinate eliminates temporal variations since it is irrelevant how the mode profile is positioned with reference to \( \varphi = 0 \). Nevertheless, each particle on the shell still undergoes motions where minima and maxima of the displacement and velocity are interchanged. If the modes are determined approximately with truncated series, the governing equation of motion can still be obtained from the minimum total energy principle, [39]

\[
\frac{\partial \Pi}{\partial \{\delta\}} = \frac{\partial E_k}{\partial \{\delta\}} + \frac{\partial E_{\delta\delta}}{\partial \{\delta\}} - \frac{\partial E_k}{\partial \{\delta\}} = \{0\}. \tag{56}
\]

Taking into account relations (45), (49) with (51), and (53) respectively one obtains the following matrix equation of natural vibrations

\[
\left( [K] + p[G]_n + \Omega^2 ([G]_k - [B]) - \omega \Omega[C] - \omega^2 [M] \right) \delta = \{0\}. \tag{57}
\]

Since Fourier coefficients \( B_0 = D_0 = F_0 = 0 \), Eqs. (41), the corresponding rows and columns in all matrices in (57) must be eliminated in order to avoid a singular eigenvalue problem.

The matrix \([C]\) multiplying the mixed \(\omega \Omega\) term, which results from the Coriolis term in the kinetic energy expression (22), is the only one causing a bifurcation of natural frequencies. The geometric stiffness matrix \([G]_k\) and the mass matrix \([B]\) are related to the centrifugal force with stiffening and softening effect respectively. The former is dominant with respect to the latter and the arithmetic mean of a bifurcated natural frequency is increased by increasing the rotational speed.

Physical meaning of the \(\delta^0\) and \(\delta^1\) terms of the strain and kinetic energy, omitted in the formulation of the eigenvalue problem, Sections 2.2 and 3.3, is analysed in Appendix E within formulation of the total rotating toroidal shell energy.

### 4.5. Axisymmetric modes

In this special case where the circumferential mode number \( n = 0 \), the shell displacements (27) are

\[
\begin{align*}
    u(\vartheta, \varphi, t) &= U(\vartheta) \cos \omega t \\
    v(\vartheta, \varphi, t) &= V(\vartheta) \sin \omega t \\
    w(\vartheta, \varphi, t) &= W(\vartheta) \cos \omega t.
\end{align*} \tag{58}
\]
With \( n = 0 \) the mode shapes are symmetric with respect to the torus axis and \( V(\vartheta) = 0 \) as well as the corresponding Fourier coefficients \( C_m \) and \( D_m \) in (42). As a result, the vector of remaining Fourier coefficients, (46), reads

\[
\langle \delta \rangle = \langle \{A_m\} \{B_m\} \{E_m\} \{F_m\} \rangle. \tag{59}
\]

Consequently, the matrices in the equation of motion (57) are reduced to

\[
[K] = \begin{bmatrix}
\begin{bmatrix} K_{11} \\ K_{31} \end{bmatrix} & \begin{bmatrix} K_{13} \\ K_{33} \end{bmatrix} \\
\begin{bmatrix} K_{13}^t \\ K_{33}^t \end{bmatrix} & \begin{bmatrix} K_{33} \end{bmatrix}
\end{bmatrix},
\]

\[
[G] = \begin{bmatrix}
\begin{bmatrix} G_{11} \\ G_{31} \end{bmatrix} & \begin{bmatrix} G_{13} \\ G_{33} \end{bmatrix} \\
\begin{bmatrix} G_{13}^t \\ G_{33}^t \end{bmatrix} & \begin{bmatrix} G_{33} \end{bmatrix}
\end{bmatrix},
\]

\[
[B] = \begin{bmatrix}
\begin{bmatrix} B_{11} \\ B_{31} \end{bmatrix} & \begin{bmatrix} B_{13} \\ B_{33} \end{bmatrix} \\
\begin{bmatrix} B_{13}^t \\ B_{33}^t \end{bmatrix} & \begin{bmatrix} B_{33} \end{bmatrix}
\end{bmatrix},
\]

\[
[M] = \begin{bmatrix}
\begin{bmatrix} M_{11} \\ M_{31} \end{bmatrix} & \begin{bmatrix} M_{13} \\ M_{33} \end{bmatrix} \\
\begin{bmatrix} M_{13}^t \\ M_{33}^t \end{bmatrix} & \begin{bmatrix} M_{33} \end{bmatrix}
\end{bmatrix},
\]

\[ \text{while } [C] = [0]. \]

As a result, there is no Coriolis force and bifurcation of natural frequencies in case of \( n = 0 \).

It is necessary to point out that the energy formulations, Eqs. (28), (30) and (33), and the resulting stiffness and mass matrices in Eq. (57) are derived for a general case of mode wave number \( n > 0 \), where \( I = \int_0^{2\pi} \cos^2(n\varphi) d\varphi = \pi \). If \( n = 0 \) (axisymmetric case) then \( I = 2\pi \) and therefore all matrices in (57) have to be multiplied by 2. This fact does not have repercussions on the solution of the eigenvalue problem (57). However, it has to be taken into account in case of forced vibrations.

5. Numerical examples

5.1. Vibration analysis of closed toroidal shell

The application of the developed numerical procedure is illustrated in the case of a closed toroidal shell with the following geometric and physical properties: \( R = 1 \text{ m}, a = 0.4 \text{ m}, h = 0.01 \text{ m}, E = 2.1 \cdot 10^{11} \text{ N/m}^2, \nu = 0.3, \) and \( \rho = 7850 \text{ kg/m}^3. \)

The first 11 natural frequencies for case \( \Omega = 0 \), determined by \( N = 5, 10 \) and 15 cosine and sine terms of Fourier series, are listed in Table 1. Results for \( N = 20 \) are equal to those of \( N = 15 \) in the first five digits. Convergence of natural frequencies is very fast, and it is enough to take only the first 10+10 terms of the Fourier series into account to achieve a reliable
solution. The free vibration problem is also solved by a 3D FEM model with ABAQUS S4R shell elements, [46]. Two FE mesh densities are used, i.e. 50×124=6200 FE and 200×500=105 FE, in order to point out convergence of the results. The final four digits are stabilized. Small discrepancies between rigorous values of natural frequencies determined by the present Rayleigh-Ritz method, N = 15, and FEM can be noticed in Table 1.

Displacements U, V and W at the shell cross-section (θ-plane in Fig. 2), for the first six natural modes are shown in Fig. 4. All three displacement components (in-plane and normal) are of the same order of magnitude. Mode profiles presented by deflection function W, take different shapes depending on wave number n in the φ-plane (Fig. 2). It is interesting that two natural modes are obtained for each n ≥ 2, with very close values of natural frequencies.

The first six natural modes obtained by the 3D FEM vibration analysis are shown in an axonometric projection and in the three orthogonal projections onto the coordinate planes in Figs. 5 and 6, respectively. Two mode profiles of the shell cross-section are obtained for each n ≥ 2, which correspond to two close natural frequencies as found by the Rayleigh-Ritz method, Table 1. Some profiles are symmetric while the others are asymmetric with respect to the y-z plane, Fig. 6. If displacement components U, V and W shown in Fig. 4 are presented as diagrams depending on angle θ, it is observed that V and W are symmetric functions, while U is antisymmetric one wit respect to θ = −π/2, for symmetric modes, and vice versa for asymmetric modes. In the top view in Fig. 6, intersections between deformed shell surface and the y-z plane are shown. In case of symmetric modes the inner intersection is a single harmonic function of the n-th order, while for asymmetric modes the intersection is described by a constant term and a harmonic of the 2n-th order.

The cross-sectional mode profiles determined by the RRM in the moving coordinate system, presented by the normal displacement, W, in Fig. 4, and those determined by FEM in the Cartesian coordinate system, presented in Fig. 6 by the total in-plane cross-sectional displacement, are quite similar. In order to make a precise comparison, the total in-plane cross-sectional displacement, \( \delta = U \hat{e}_i + W \hat{e}_3 \), obtained by the RRM and the total in-plane cross-sectional displacement, \( \delta = \delta_x \hat{i} + \delta_z \hat{k} \), resulting from FEM are shown in Fig. 7a for mode number 4. The intersections between deformed shell and the y-z plane determined by RRM and FEM are compared in Fig. 7b. It is obvious that the mode shapes generated by RRM and FEM are very similar in spite of their rather complicated asymmetric form.
Next, the considered toroidal shell is exposed to internal pressure. The geometric stiffness is now increased due to the membrane pre-stress, Eqs. (40). The obtained natural frequencies for the first five modes are listed in Table 2 and compared with the FEM results determined by ABAQUS, [46]. A very good agreement is obtained, since the membrane force and total force $N_{1p}$ are almost the same, while the total force $N_{2p}$, which is cca. one half of the average $N_{1p}$, varies around the value calculated according to the membrane theory with $\pm 9\%$, Eqs. (40). It means that the membrane forces are dominant in the total tension forces used in the FEM analysis.

Vibration analysis of the rotating toroidal shell is considered next. The quadratic eigenvalue problem, Eq. (57), is solved for different values of rotational speed, $\Omega$. The obtained values for natural frequencies, for $n = 0, 2$ and $3$ are shown in Figs. 8 and 9 for asymmetric and symmetric modes, respectively, as a function of dimensionless rotational speed $\Omega/\omega_0$, where $\omega_0 = 80.73$ Hz, Table 1. In case of $n = 0$ (axisymmetric mode) there is no bifurcation of natural frequencies since matrix $[C]$ in Eq. (57) is zero. Moreover, it can be shown analytically that matrix $[G]\xi$ is identical to $[B]$ if $n = 0$, and therefore there is no influence of centrifugal force on natural frequencies. Bifurcation occurs for $n \geq 2$ and natural frequencies for forward and backward rotating modes result that are either positive or negative, respectively. The absolute values of frequencies corresponding to forward rotating natural modes are lower than those corresponding to backward natural modes.

The same analysis is also carried out by FEM, [46], in the fixed coordinate system. The obtained results are converted into the co-rotating system, and they are included in Figs. 8 and 9 for comparison, where circles in diagrams denote distinct values of $\Omega/\omega_0$ for which the calculation is performed. Some differences between natural frequencies determined by RRM and FEM can be noticed. This is mainly caused by different formulation of tension forces $N_{\theta}$ and $N_{\phi}$ in the geometric stiffness matrix $[G]\xi$ in Eq. (57) and in ABAQUS. In RR method analytical expressions (40) for tension forces based on the membrane theory are used, while in the FEM analysis they are obviously determined numerically by employing the shell theory. Therefore, the total tension forces (membrane + bending) are also calculated by the RR method using the shell theory approach as described in Appendix F. The total displacements of the shell cross-section are shown in Fig. 10. The RRM and FEM results are almost the same. The membrane and total tension forces $N_{\phi}$ and $N_{\theta}$ are shown in Fig. 11 for rotational speed $\Omega = 60$ rad/s. The latter force is very small comparing to the former (cca.
The total tension forces agree very well with those determined by FEM so that their difference is indistinguishable in Fig. 11. However, there is a large difference between membrane and total tension forces. The membrane forces are determined directly from the equilibrium equations, i.e. on the basis of the undeformed geometry, Section 3.4. In the shell theory the equilibrium equation in terms of displacements are used, i.e. the tensional forces are calculated on the basis of the deformed geometry taking into account tensional and bending stiffness. As a result, the change in the shell geometry due to mean stresses, has a noticeable influence on natural frequencies.

Calculation of natural vibrations of the rotating toroidal shell by the RR method is now repeated taking into account values of the total pretension forces. The obtained natural frequencies for n=0,2 and 3 are included in Figs. 8 and 9. It can be seen that FEM results for n = 0 are bounded by values obtained by RRM-ST (Shell Theory) and RRM-MT (Membrane Theory), Fig. 8. For n = 2 and 3 FEM results are nearer to either RRM-ST or RRM-MT values. It is necessary to mention that ABAQUS operates with logarithmic nonlinear strains, [46]. The values of natural frequencies for n=2 are listed in Table 3.

In order to investigate the influence of the engineering strains on natural vibrations of the rotating toroidal shell, Section 2.2, the calculation is performed with corresponding coefficients of the geometric stiffness matrix, c_i(q), i = 1,2…11 specified in Appendix B. The obtained results for n=2 are listed in Table 4. Comparing values from Table 4 with those in Table 3, determined by employing the Green-Lagrange strains, some differences can be noticed. Values of natural frequencies determined with the engineering strains, are included in Figs. 8 and 9 only at \( \Omega/\omega_0 = 1 \) and marked with arrows, in order to avoid clustering of many lines. Natural frequencies determined by employing the engineering strains are somewhat higher than those obtained with the Green-Lagrange strain formulation. Generally speaking, a somewhat better agreement of the RRM results, determined by the Green-Lagrange strains, with the FEM results is obtained.

5.2. Vibration analysis of thin-walled toroidal ring

A thin-walled toroidal ring of the following geometric and physical properties is considered: \( R = 1 \) m, \( a = 0.05 \) m, \( h = 0.001 \) m, \( E = 2.1 \cdot 10^{11} \) N/m², \( \nu = 0.3 \), \( \rho = 7850 \) kg/m³. Natural frequencies of the first four modes are listed in Table 5 and compared with those obtained by FEM analysis. A very good agreement can be seen. The corresponding natural modes are shown in Figs. 12 and 13. For each value of n, in-plane and out-of-plane flexural
modes are obtained. In spite of the fact that the shell is similar to a ring, \( a/R = 0.05 \), deformations of cross-sections occur due to a relatively small thickness, \( h/a = 0.02 \).

Natural frequencies of the rotating ring are also determined considering the total tension forces calculated according to the shell theory for \( n = 2 \) and 3. The obtained values for total and membrane tension forces, Eqs. (40), are compared with the FEM results in Figs. 14 and 15, and quite a good agreement from an engineering point of view is achieved. Bifurcation of natural frequencies of out-of-plane vibrations is very small. In case of a ring with a solid cross-section, bifurcations of natural frequencies of out-of-plane vibration modes do not occur [13].

Values of natural frequencies determined for the engineering strain at \( \Omega/\omega_0 = 1 \) are marked in Figs. 14 and 15 with arrows. Somewhat better agreement with the FEM results is obtained if the Green-Lagrange formulation of nonlinear strain is used in RRM.

5.3. On mode symmetry and antisymmetry

In this study the geometry of a toroidal shell is described in the conventional coordinate system ordinarily used for shells of revolution, [1]-[6]. The displacements of the shell cross-section, \( U(\vartheta) \), \( V(\vartheta) \) and \( W(\vartheta) \), Eqs. (41), are assumed in the form of the full Fourier series. However, either symmetric or antisymmetric modes with respect to the symmetry line \( (\vartheta = \pi/2) \), Fig. 6, are obtained. Symmetric modes can be easily recognized, while antisymmetric ones cannot be seen easily. Nevertheless, if the distribution of displacements as a function of the meridional coordinate \( \vartheta \) are presented by diagrams normalized with \( W_{\text{max}} \), then the mode antisymmetry is evident, Figs. 16 and 17. This fact is confirmed by zero values of every second Fourier coefficient, as illustrated in Fig. 18. These figures also indicate a fast convergence of the Fourier coefficients whose amplitudes decay quickly with the increase of \( m \).

Accordingly, one can write for symmetric modes

\[
U(\vartheta) = \sum_{m=1,3,5\ldots}^{\infty} A_m \cos m\vartheta + \sum_{m=2,4\ldots}^{\infty} B_m \sin m\vartheta \\
V(\vartheta) = \sum_{m=0,2,4\ldots}^{\infty} C_m \cos m\vartheta + \sum_{m=1,3,5\ldots}^{\infty} D_m \sin m\vartheta \\
W(\vartheta) = \sum_{m=0,2,4\ldots}^{\infty} E_m \cos m\vartheta + \sum_{m=1,3,5\ldots}^{\infty} F_m \sin m\vartheta
\] (61)

and for antisymmetric modes
Mode symmetry and antisymmetry can be used to simplify the procedure of vibration analysis of a closed toroidal shell. However, it is necessary to change the conventional coordinate system, Fig. 2, so that \( \vartheta = \eta + \pi/2 \), where \( \eta \) is the meridional angle measured from the torus symmetry plane. In such a case all trigonometric functions in the variable coefficients, Appendices A and B, must be exchanged accordingly: \( \sin \vartheta = \cos \eta \), \( \cos \vartheta = -\sin \eta \). In this case the shell displacements are described by reduced Fourier series.

For symmetric modes:

\[
U(\eta) = \sum_{m=0}^{\infty} B_m \sin m\eta \\
V(\eta) = \sum_{m=0}^{\infty} C_m \cos m\eta \\
W(\eta) = \sum_{m=0}^{\infty} E_m \cos m\eta
\]

For antisymmetric modes:

\[
U(\eta) = \sum_{m=0}^{\infty} A_m \cos m\eta \\
V(\eta) = \sum_{m=1}^{\infty} D_m \sin m\eta \\
W(\eta) = \sum_{m=1}^{\infty} F_m \sin m\eta
\]

Consequently, all submatrices \([k]_i\), Appendix C, are reduced accordingly. In order to simplify the numerical procedure it is convenient to reduce system of equations (57), taking into account that for symmetric modes

\[
\langle \delta \rangle = \langle \{B_m\} \{C_m\} \{E_m\}\rangle
\]

and for antisymmetric modes

\[
\langle \delta \rangle = \langle \{A_m\} \{D_m\} \{F_m\}\rangle.
\]
5.4. Applicability domain of the proposed linearized method

The proposed method deals with linearized geometrically nonlinear problem. In order to analyse its applicability, natural frequencies of the toroidal shell, Section 5.1, are determined for the symmetric mode 3, \( n = 2 \), Table 2, Figs. 5 and 6, by varying the internal pressure up to an extremely high value of \( 10^3 \) MPa. The same problem is solved by the commercial software CATIA for linear analysis and ABAQUS for nonlinear analysis. The obtained results for all three cases are shown in Fig. 19 in the logarithmic scale. It can be observed that up to pressure of 100 MPa the difference between linear and nonlinear solutions is very small.

Internal pressure is actually limited in real shell structures by permissible stress criterion for tensional forces. In the considered case of the closed toroidal shell the membrane forces are dominant. The meridional force achieves maximum value, Eqs. (40), at point \( \psi = -\pi / 2 \), Fig. 2. Hence the maximum stress reads

\[
\sigma_{p\text{max}} = \frac{pa}{2h} \frac{2R-a}{R-a}.
\]

(67)

If it is assumed that the maximum stress is limited by a yielding stress, \( R_e \), for the ultimate pressure one obtains

\[
p_u = \frac{2h}{a} \frac{R-a}{2R-a} R_e.
\]

(68)

Values of ultimate pressure for the toroidal shell made of ordinary steel, and two high tensile steels of different quality, are listed in Table 6. It is obvious that even the highest pressure value is within domain of the linear shell dynamic behaviour, Fig 19. Natural frequencies of the shell exposed to the pressure of \( 10^3 \) MPa in the case of the Green-Lagrange strains and engineering strains read 697 Hz and 681 Hz, respectively, Fig. 19.

In a similar way the ultimate shell rotation speed can be determined. In this case the circumferential membrane force, Eqs. (40), takes maximum value at point \( \psi = -\pi / 2 \), Fig. 2, and the maximum stress is

\[
\sigma_{2\Omega\text{max}} = \rho \left( R + a \right)^2 \Omega^2.
\]

(69)

Equalling the maximum stress to the yielding stress, gives

\[
\Omega_u = \frac{1}{R + a} \sqrt{\frac{R_e}{\rho}}.
\]

(70)
Values of $\Omega_u$ for the steel qualities are listed in Table 6, and expressed in dimensionless form $\Omega/\omega_0$. By this parameter it is possible to enter into the diagrams of bifurcated natural frequencies of asymmetric and symmetric natural modes, Figs. 8 and 9, respectively. It can be observed that domain of rotational speed $0 \leq \Omega/\omega_0 \leq 1$ is still linear, and there are no large differences between the natural frequencies determined by linear and nonlinear methods.

It is worthwhile to mention that in the considered example of steel toroidal shell, the change of toroidal geometry due to the ultimate pre-stressing by the centrifugal load is comparatively small, of order $W/a = 10^{-3}$, Fig. 10. However, a parameter $W/h = 0.04$ is relevant for assessing the contribution of bending to the tensional forces.

6. Conclusion

In this self-contained paper vibrations of pressurised rotating toroidal shells with closed cross-section are analysed by the Rayleigh-Ritz method. Fourier series are used to describe the displacement components as a function of the meridional coordinate whereas their dependence on the circumferential coordinate is described exactly using convenient trigonometric sine and cosine functions. Linear strain-displacement relationships, the ordinary strain energy, and the kinetic energy are derived from general expressions for thin shells of revolution. For the strain energy due to pre-stressing, the Green-Lagrange nonlinear strain-displacement relation is employed. Pre-stressing tension forces due to the internal pressure and centrifugal load are derived by employing the membrane and the shell theory. Numerical examples show that tension forces determined according to the membrane assumption are very close to those determined by the shell theory (membrane + bending), in case of internal pressure. However, with centrifugal load there are some differences between the tension forces calculated according to the membrane theory and the total tension forces calculated according to the shell theory.

The developed procedure for vibration analysis of toroidal shells by employing Rayleigh-Ritz method and Fourier series is rather complicated. Ordinary stiffness matrix, geometric stiffness matrix and mass matrices, related to the pressurisation and centrifugal loads, Coriolis force and inertia load, depend on a large number of variable coefficients and submatrices. In spite of this, the procedure is presented in a consistent and physically transparent way, which is also easy for computer coding. The quadratic eigenvalue problem, \[ \text{Det}\left[K\left(\omega^0, \omega', \omega^2\right)\right]_{bn} = 0, \] is solved by a commercial package as a polynomial eigenvalue problem.
problem, [48]. Forward and backward modes rotating in the circumferential direction and the corresponding natural frequencies are obtained.

The derived Coriolis mass matrix by employing the Rayleigh-Ritz method, and assuming displacement field by Fourier series, is symmetric, while in the finite element formulation the antisymmetric matrix is obtained.

The presented procedure based on the Rayleigh-Ritz method and Fourier series, is semi-analytical. The convergence of results is very fast. Only 15 sine and cosine terms of the three sets of Fourier series for displacements is sufficient to achieve accurate results. The number of equations in the eigenvalue problem is 90, while the FEM analysis, depending on mesh density 50×124 and 200×500, related to the wanted accuracy, includes 3.7×10^4 d.o.f. and 6×10^5 d.o.f., respectively. The CPU time in the above three cases is about 7 seconds (RRM), 81 s (50×124), and 420 s (200×500). Hence, savings of the time consumption by applying the proposed method is considerable. Also, there is some time saving due to the shell geometry modelling by RRM with respect to FEM in particular if parametric studies are required.

Vibrations of two characteristic toroidal shells are analysed, i.e. one of an ordinary ratio of geometric parameters and the other one which can be seen as a thin-walled toroidal ring. In both examples two distinctive spectra of natural frequencies are obtained. In the first example they are related to symmetric and asymmetric natural modes, respectively. In the second example typical in-plane and out-of-plane natural modes of the ring are recognized. These two numerical examples can be used as a benchmark for evaluation of numerical methods.

Within the numerical examples influence of the Green-Lagrange nonlinear strains and engineering strains on the response of rotating toroidal shell is investigated. It is found that somewhat better agreement with FEM results is obtained if the Green-Lagrange second order strain-displacement relation is used.

The presented semi-analytical method deals with linearized geometrically nonlinear problem and is applicable in domain of linear toroidal shell dynamic behaviour. In the considered case this domain considerably exceeds the pre-stressing limit given by the yielding stress. A change of the toroidal shell cross-section geometry due to pre-stressing is consistently taken into account. The eigenvalue problem is formulated free of the static displacements. Their influence is taken into account via the tensional forces. As a result, by solving the eigenvalue problem, the bifurcated natural and pure natural modes are obtained.
Shell vibrates with respect to the statically deformed geometry. The obtained results are rigorous in the linear domain.

In further investigations the same energy approach will be used for development of the finite strip method, like that already worked out for cylindrical shell, [19]. This will enable to analyse vibration problems of a toroidal shell with open cross section, or shell structures of revolution with a toroidal segment like for instance automotive tires. Moreover, by such a finite strip with adaptive curvature it will be possible to analyse vibrations of a toroidal shell made of some more elastic material with large pre-stressing deformations, resulting in a pseudo-toroidal shell with an irregular configuration, as a geometrically nonlinear problem.

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References


Appendix A

Variable coefficients of the strain energy

\[
p_1 = \pi \left( K + \frac{D}{a^2} \right) \frac{r}{a}
\]
\[
p_2 = \pi \left( K + \frac{D}{a^2} \right) \frac{a}{r} \left[ \cos^2 \vartheta + \frac{1}{2} (1 - \nu) n^2 \right]
\]
\[
p_3 = \pi \left( K + \frac{D}{a^2} \right) \nu \cos \vartheta
\]
\[
p_4 = \frac{1}{2} \pi (1 - \nu) \left( K + \frac{D}{r^2} \sin^2 \vartheta \right) \frac{r}{a}
\]
\[
p_5 = \pi \frac{a}{r} \left\{ K \left[ n^2 + \frac{1}{2} (1 - \nu) \cos^2 \vartheta \right] + \frac{D}{r^2} \left[ n^2 \sin^2 \vartheta + \frac{1}{2} (1 - \nu) \left( \frac{r}{a} \right)^2 \cos^2 \vartheta \left( 1 - 2 \frac{a}{r} \sin \vartheta \right)^2 \right] \right\}
\]
\[
p_6 = \frac{1}{2} \pi (1 - \nu) \cos \vartheta \left[ -K + \frac{D}{ar} \sin \vartheta \left( 1 - 2 \frac{a}{r} \sin \vartheta \right) \right]
\]
\[
p_7 = \pi \nu n \left( K + \frac{D}{ar} \sin \vartheta \right)
\]
\[
p_8 = -\frac{1}{2} \pi (1 - \nu) n \left( K + \frac{D}{ar} \sin \vartheta \right)
\]
\[
p_9 = \frac{1}{2} \pi n \cos \vartheta \left\{ K (3 - \nu) \frac{a}{r} + \frac{D}{r^2} \left[ 2 \sin \vartheta - (1 - \nu) \frac{r}{a} \left( 1 - 2 \frac{a}{r} \sin \vartheta \right) \right] \right\}.
\]
\[ q_1 = \pi \frac{D}{a^2} \frac{r}{a} \]
\[ q_2 = \pi \frac{D}{ar} \left[ \cos^2 \vartheta + 2(1 - \nu)n^2 \right] \]
\[ q_3 = \pi \left\{ K \left( \frac{r + a}{r} \sin^2 \vartheta + 2\nu \sin \vartheta \right) + \frac{D}{r^2} \frac{a}{r} n^2 \left[ n^2 + 2(1 - \nu)\cos^2 \vartheta \right] \right\} \]
\[ q_4 = \pi \nu \frac{D}{a^2} \cos \vartheta \]
\[ q_5 = -\pi \nu \frac{D}{r} n^2 \cos \vartheta \]
\[ q_6 = -\pi (3 - 2\nu) \frac{D}{r^2} n^2 \cos \vartheta \]
\[ q_7 = -\pi \frac{D}{a^2} \frac{r}{a} \]
\[ q_8 = -\pi \nu \frac{D}{a^2} \cos \vartheta \]
\[ q_9 = -\pi \frac{D}{ar} \left[ \cos^2 \vartheta + (1 - \nu)n^2 \right] \]
\[ q_{10} = \pi \left[ K \left( \frac{r}{a} \left( 1 + \nu \frac{a}{r} \sin \vartheta \right) + \nu \frac{D}{ar} n^2 \right) \right] \]
\[ q_{11} = \pi \left[ K \left( \frac{a}{r} \sin \vartheta + \nu \right) \cos \vartheta + (2 - \nu) \frac{D}{r^2} n^2 \cos \vartheta \right] \]
\[ q_{12} = -\pi \nu \frac{D}{ar} n \sin \vartheta \]
\[ q_{13} = \pi (1 - \nu) \frac{D}{ar} n \sin \vartheta \]
\[ q_{14} = \pi \frac{D}{r^2} n \cos \vartheta \left[ (1 - \nu) \frac{r}{a} \left( 1 - 2 \frac{a}{r} \sin \vartheta \right) - \sin \vartheta \right] \]
\[ q_{15} = -\pi (1 - \nu) \frac{D}{r^2} n \sin \vartheta \cos \vartheta \]
\[ q_{16} = \pi n \left\{ K \left( \frac{a}{r} \sin \vartheta + \nu \right) + \frac{D}{r^2} \left[ \frac{a}{r} n^2 \sin \vartheta - (1 - \nu) \cos^2 \vartheta \left( 1 - 2 \frac{a}{r} \sin \vartheta \right) \right] \right\} \]  \quad \text{(A2)}
Appendix B

Variable coefficients of the strain energy due to pre-stressing

1. Green-Lagrange strains

\[ c_1 = c_3 = c_6 = c_9 = \pi \frac{r}{a} N_\theta \]
\[ c_2 = \pi \left[ \frac{r}{a} N_\theta + \frac{a}{r} (n^2 + \cos^2 \theta) N_\phi \right] \]
\[ c_4 = \pi \frac{a}{r} \left[ \cos^2 \theta N_\theta + (n^2 + \sin^2 \theta) N_\phi \right] \]
\[ c_5 = -\pi \cos \theta N_\theta \]
\[ c_7 = \pi \left[ \frac{r}{a} N_\theta + \frac{a}{r} (n^2 + \sin^2 \theta) N_\phi \right] \]
\[ c_8 = \pi \frac{a}{r} n \cos \theta N_\phi \]
\[ c_{10} = \pi \frac{a}{r} \sin \theta \cos \theta N_\phi \]
\[ c_{11} = 2\pi \frac{a}{r} n \sin \theta N_\phi . \] (B1)

2. Engineering strains

\[ c_1 = 0, \quad c_3 = c_6 = c_9 = \pi \frac{r}{a} N_\theta \]
\[ c_2 = \pi \left( \frac{r}{a} N_\theta + \frac{a}{r} n^2 N_\phi \right) \]
\[ c_4 = \pi \frac{a}{r} \left( \cos^2 \theta N_\theta + \sin^2 \theta N_\phi \right) \]
\[ c_5 = -\pi \cos \theta N_\theta \]
\[ c_7 = \pi \frac{a}{r} n^2 N_\phi \]
\[ c_8 = c_{10} = c_{11} = 0. \] (B2)
Appendix C

Submatrices of the stiffness matrix

\[
[k]_k = \begin{bmatrix} f_i f_m & f_i g_m \\ g_i f_m & g_i g_m \end{bmatrix} \\
[k]_b = \begin{bmatrix} f_i f_m & f_k g_m \\ g_i f_m & g_k g_m \end{bmatrix} \\
[k]^\text{p}_i = \begin{bmatrix} f_i f_m & f_k g_m \\ g_i f_m & g_k g_m \end{bmatrix} \\
[k]^\text{p}_k = \begin{bmatrix} f_i f_m & f_i g_m \\ g_i f_m & g_i g_m \end{bmatrix} \\
[k]^\text{p}_b = \begin{bmatrix} f_i f_m & f_k g_m \\ g_k f_m & g_k g_m \end{bmatrix} \]

(C1)

\[
[k]_b = [k]^\text{p}_i + [k]^\text{p}_k \\
[k]_k = [k]^\text{p}_i + [k]^\text{p}_k \\
[k]_b = [k]^\text{p}_b + [k]^\text{p}_b
\]
Appendix D

Submatrices of the mass matrices

\[
[B]_{11} = \alpha \int_0^{2\pi} r \cos^2 \vartheta [k]^2 \, d\vartheta \\
[B]_{22} = \alpha \int_0^{2\pi} r[k]_2 \, d\vartheta \\
[B]_{33} = \alpha \int_0^{2\pi} r \sin^2 \vartheta [k]^2 \, d\vartheta \\
[B]_{13} = \alpha \int_0^{2\pi} r \sin \vartheta \cos \vartheta [k]_2 \, d\vartheta \\
[B]_{11} = [B]_{33}^T
\]

\[
[C]_{12} = 2\alpha \int_0^{2\pi} r \cos \vartheta [k]_2 \, d\vartheta \\
[C]_{22} = 2\alpha \int_0^{2\pi} r \sin \vartheta [k]_2 \, d\vartheta \\
[C]_{21} = [C]_{12}^T, \quad [C]_{32} = [C]_{23}^T
\]

\[
[M]_{11} = [M]_{22} = [M]_{33} = \alpha \int_0^{2\pi} r[k]_2 \, d\vartheta \\
\alpha = \pi \rho \varphi a.
\]
Appendix E
Lower order strain and kinetic energy terms

According to the classification of the ordinary strain energy, $E_S$, strain energy due to pre-stressing, $E_G$, and kinetic energy, $E_k$, in terms of different orders of dynamic displacements, Section 2.1, 3.2 and 3.3, one can write differences between the strain and kinetic energies in symbolic form

$$\Pi = \left[ E_G \left( \delta^0 \right) - E_k \left( \delta^0 \right) \right] + \left[ E_G \left( \delta^1 \right) - E_k \left( \delta^1 \right) \right] + \left[ E_S \left( \delta^2 \right) + E_G \left( \delta^2 \right) - E_k \left( \delta^2 \right) \right] + O \left[ E_G \left( \delta^3 \right) + E_G \left( \delta^4 \right) \right].$$  \hspace{1cm} (E1)

Expanding Eq. (12) for a toroidal shell, and following the procedure from Section 2.2, one obtains the zero order strain energy in the form of Eq. (13), taking $\varepsilon_{11}^0$ and $\varepsilon_{22}^0$ instead of $\varepsilon_{11}^*$ and $\varepsilon_{22}^*$ into account. Using relations (10), yields

$$E_G \left( \delta^0 \right) = \frac{1}{2Eh} \int_0^{2\pi} \int_0^{2\pi} \left( N_1^2 + N_2^2 - 2\nu N_1 N_2 \right) \rho \, d\theta d\varphi. \hspace{1cm} (E2)$$

In order to investigate the particular contribution of the shell rotation, the internal pressure is omitted in Eqs. (40). Thus $N_1 = 0$ and $N_2 = \rho hr^2 \Omega^2$. Strictly speaking, radius $r$, Fig. 2, is increased due to static displacements $u_0$ and $w_0$ caused by centrifugal load and determined by the shell theory

$$\bar{R} = r + u_0 \cos \theta + w_0 \sin \theta. \hspace{1cm} (E3)$$

Taking into account the above facts one finds after integration of Eq. (E2) per $\varphi$

$$E_G \left( \delta^0 \right) = \frac{\pi a}{Eh} \rho^2 h^2 \Omega^4 \int_0^{2\pi} \bar{R}^4 \, d\varphi. \hspace{1cm} (E4)$$

From expansion of Eq. (12) the first order strain energy is obtained in the form

$$E_G \left( \delta^1 \right) = \int_0^{2\pi} \int_0^{2\pi} \left( \varepsilon_{11}^* N_1 + \varepsilon_{22}^* N_2 \right) \rho \, d\theta d\varphi, \hspace{1cm} (E5)$$

which is similar to that of the second order formula (15). Substituting $N_1 = 0$ and $N_2 = \rho h \Omega^2 \bar{R}^2$, and formula $\varepsilon_{22} = \varepsilon_{\varphi}$ from (24) expressed in terms of displacements defined with Eqs. (27), into (E5), one arrives after integration per $\varphi$, at
\[ E_g(\delta^i) = 2\pi \rho h a \Omega^2 \int_0^{2\pi} (U \cos \vartheta + W \sin \vartheta) \tilde{R}r d\vartheta \cos \omega t. \] (E6)

The only terms that are left after integration, related to the stationary modes, are those of the mode number \( n = 0 \).

On the other hand, one finds from Eq. (32), taking into account (31), the following expression for the zero order kinetic energy

\[ E_k(\delta^0) = \frac{1}{2} \rho h \Omega^2 \int_0^{2\pi} \int_0^{2\pi} (r + u_0 \cos \vartheta + w_0 \sin \vartheta)^2 \text{rad}r d\vartheta. \] (E7)

Furthermore, using Eq. (E3) and after integrating (E7) per \( \varphi \), yields

\[ E_k(\delta^0) = \pi \rho h a \Omega^2 \int_0^{2\pi} \tilde{R}^2 r d\vartheta. \] (E8)

In a similar way, one finds from Eq. (32)

\[ E_k(\delta^1) = \rho h a \Omega^2 \int_0^{2\pi} \int_0^{2\pi} \left[ \dot{V} + (u \cos \vartheta + w \sin \vartheta) \Omega \right] \tilde{R}r d\vartheta d\varphi. \] (E9)

Substituting Eqs. (27) for displacements into (E9), and after integration per \( \varphi \), one obtains the expression identical to (E6). Displacement \( V \) vanishes from (E9) since \( n = 0 \). As a result \( E_g(\delta^1) - E_k(\delta^1) = 0 \) and the expression in the second brackets in (E1) disappears.

The zero order strain energy and kinetic energies represent the accumulated energy in the shell due to action of centrifugal load.

\[ E^0_{\text{str}} = E_g(\delta^0) + E_k(\delta^0) = \pi \rho h a \Omega^2 \left[ \frac{\rho}{E} \Omega^2 \int_0^{2\pi} \tilde{R}^3 r d\vartheta + \int_0^{2\pi} \tilde{R}^2 r d\vartheta \right]. \] (E10)

The strain energy grows more rapidly by increasing rotation speed than the kinetic energy. It is obvious that \( E_g(\delta^0) - E_k(\delta^0) \) is constant for given \( \Omega \), and vanishes in the variation of \( \Pi \) per dynamic displacements, Eq. (E1).

Based on the above consideration, only the expression in the third brackets of Eq. (E1) of the \( \delta^2 \)-order remains for natural vibration analysis. Strain and kinetic energies indirectly depend on static displacements via tensional forces. Thus, the eigenvalue problem solution results with bifurcated natural frequencies and pure natural modes. According to Eqs. (31), shell vibrates with respect to the statically deformed geometry.
Appendix F
Determination of tension forces due to centrifugal load

Axisymmetric deformation is assumed: \( n = 0, \ u = U, \ v = V = 0, \ w = W \). The total energy consists of the strain energy \( E_s(U,W) \), Eq. (28), and the work of centrifugal load \( q_i \) and \( q_n \), Eqs. (35). Due to the energy balance, the difference of these energies has to be zero in the case of exact solution, and has to take minimum value in an approximate solution. Hence, one can write

\[
\Pi = E_s - W_q,
\]

where

\[
W_q = \int_0^{2\pi} \int_0^2 (q_i U + q_n W) r a \vartheta d\vartheta d\varphi = 2\pi a \int_0^{2\pi} (q_i U + q_n W) r d\vartheta.
\]

Substituting Eqs. (35) for load and Eqs. (42) for displacements into (F2), the derivatives of the load work per Fourier coefficients read

\[
\frac{\partial W_q}{\partial \{\delta\}} = \{F\},
\]

where

\[
\langle \delta \rangle = \langle \langle A_m \rangle B_m \rangle E_m F_m \rangle
\]

\[
\langle F \rangle = 2\pi\rho h a\Omega^2 \langle I_{k1}, I_{k2}, I_{k3}, I_{k4} \rangle.
\]

\[
I_{k1} = \int_0^{2\pi} r^2 \cos k\vartheta \cos \vartheta d\vartheta
\]

\[
I_{k2} = \int_0^{2\pi} r^2 \sin k\vartheta \cos \vartheta d\vartheta
\]

\[
I_{k3} = \int_0^{2\pi} r^2 \cos k\vartheta \sin \vartheta d\vartheta
\]

\[
I_{k4} = \int_0^{2\pi} r^2 \sin k\vartheta \sin \vartheta d\vartheta.
\]

Using the principle of minimum total energy, [39], \( i.e. \) minimum error

\[
\frac{\partial \Pi}{\partial \{\delta\}} = \frac{\partial E_s}{\partial \{\delta\}} - \frac{\partial W_q}{\partial \{\delta\}} = \{0\}
\]
and relation (45), a system of non-homogenous algebraic equations is obtained

\[ 2[K]\{\delta\} = \{F\}. \]  

(F8)

The stiffness matrix, [K], reduced to the form (60), is multiplied by 2, for reasons explained at the end of Section 4.5.

Since Fourier coefficients \( B_0 = F_0 = 0 \), the corresponding equations are excluded from the matrix equation (F8). For coefficients in Fourier series characterised by \( k = 1 \), two identical equations are obtained in (F8) for \( B_1 \) and \( F_1 \), as can be seen in the load terms, Eq. (F6). Therefore, one of these equations is omitted in order to avoid singularity of the stiffness matrix.

By calculating displacements \( U \) and \( W \) it is possible to determine the tension strains, Eqs. (24), as well as the tension forces by employing Hooke's law

\[
N_\theta = \frac{Eh}{1-\nu^2} \left[ \frac{1}{a} \frac{dU}{d\theta} + \frac{\nu}{r} U \cos \theta + \left( \frac{1}{a} + \frac{\nu}{r} \sin \theta \right) W \right],
\]

\[
N_\varphi = \frac{Eh}{1-\nu^2} \left[ -\frac{1}{r} U \cos \theta + \frac{\nu}{a} \frac{dU}{d\theta} + \left( \frac{1}{r} \sin \theta + \frac{\nu}{a} \right) W \right].
\]  

(F9)
Table 1. Convergence of natural frequencies of toroidal shell, $\omega$ [Hz], $R = 1$ m, $a = 0.4$ m, $h = 0.01$ m

<table>
<thead>
<tr>
<th>Mode no.</th>
<th>Mode type</th>
<th>$n$</th>
<th>Number of Fourier terms, $N$</th>
<th>FEM 50×124 FE</th>
<th>FEM 200×500 FE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>5</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>1</td>
<td>Asym.</td>
<td>0</td>
<td>89.37</td>
<td>80.73</td>
<td>80.73</td>
</tr>
<tr>
<td>2</td>
<td>Asym.</td>
<td>2</td>
<td>133.06</td>
<td>111.12</td>
<td>111.11</td>
</tr>
<tr>
<td>3</td>
<td>Sym.</td>
<td>2</td>
<td>148.25</td>
<td>123.05</td>
<td>123.05</td>
</tr>
<tr>
<td>4</td>
<td>Asym.</td>
<td>3</td>
<td>247.50</td>
<td>207.40</td>
<td>207.40</td>
</tr>
<tr>
<td>5</td>
<td>Sym.</td>
<td>3</td>
<td>248.16</td>
<td>207.86</td>
<td>207.85</td>
</tr>
<tr>
<td>6</td>
<td>Sym.</td>
<td>4</td>
<td>369.69</td>
<td>309.75</td>
<td>309.74</td>
</tr>
<tr>
<td>7</td>
<td>Asym.</td>
<td>4</td>
<td>369.95</td>
<td>309.90</td>
<td>309.89</td>
</tr>
<tr>
<td>8</td>
<td>Asym.</td>
<td>1</td>
<td>359.75</td>
<td>351.06</td>
<td>351.06</td>
</tr>
<tr>
<td>9</td>
<td>Asym.</td>
<td>2</td>
<td>412.49</td>
<td>398.62</td>
<td>398.61</td>
</tr>
<tr>
<td>10</td>
<td>Sym.</td>
<td>2</td>
<td>413.74</td>
<td>401.29</td>
<td>401.28</td>
</tr>
<tr>
<td>11</td>
<td>Sym.</td>
<td>1</td>
<td>429.62</td>
<td>415.23</td>
<td>415.22</td>
</tr>
</tbody>
</table>

Table 2. Natural frequencies of toroidal shell exposed to internal pressure, $\omega$ [Hz], $R = 1$ m, $a = 0.4$ m, $h = 0.01$ m, $N=15$

<table>
<thead>
<tr>
<th>$p$ (bar)</th>
<th>$\omega_1$ $n = 0$</th>
<th>$\omega_2$ $n = 2$</th>
<th>$\omega_3$ $n = 2$</th>
<th>$\omega_4$ $n = 3$</th>
<th>$\omega_5$ $n = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>80.73 (80.87)*</td>
<td>111.11</td>
<td>123.05</td>
<td>207.40</td>
<td>207.85</td>
</tr>
<tr>
<td>20</td>
<td>98.54 (97.22)</td>
<td>131.21</td>
<td>144.96</td>
<td>237.51</td>
<td>238.02</td>
</tr>
<tr>
<td>40</td>
<td>113.18 (110.66)</td>
<td>147.77</td>
<td>163.11</td>
<td>263.54</td>
<td>263.88</td>
</tr>
<tr>
<td>60</td>
<td>125.81 (122.22)</td>
<td>161.88</td>
<td>178.77</td>
<td>286.05</td>
<td>286.68</td>
</tr>
<tr>
<td>80</td>
<td>137.06 (132.46)</td>
<td>174.22</td>
<td>192.59</td>
<td>306.64</td>
<td>307.15</td>
</tr>
<tr>
<td>100</td>
<td>147.26 (141.70)</td>
<td>185.38</td>
<td>205.02</td>
<td>325.04</td>
<td>325.78</td>
</tr>
</tbody>
</table>

* FEM-ABAQUS, 50×124 FE
Table 3. Natural frequencies of rotating toroidal shell, $\omega$ [Hz], $R = 1$ m, $a = 0.4$ m, $h = 0.01$ m, $n = 2$, $\omega_b = 80.73$ Hz, ($N=20$), Green-Lagrange strains

| $\Omega/\omega_b$ | $\Omega$ [Hz] | Asymmetric | | Symmetric | |
|-------------------|---------------|------------|----------------|----------------|
|                   |               | Forward $n = -2$ | Backward $n = 2$ | Forward $n = -2$ | Backward $n = 2$ |
| 0                 | 0.0           | 111.11     | 111.11         | 123.05         | 123.05         |
| 0.1               | 8.073         | 107.98     | 115.94         | 120.58         | 127.30         |
| 0.2               | 16.146        | 106.50     | 122.43         | 119.87         | 133.31         |
| 0.3               | 24.219        | 106.56     | 130.48         | 120.78         | 140.99         |
| 0.4               | 32.292        | 108.01     | 139.93         | 123.13         | 150.16         |
| 0.5               | 40.365        | 110.64     | 150.60         | 126.72         | 160.64         |
| 0.6               | 48.438        | 114.27     | 162.30         | 131.35         | 172.24         |
| 0.7               | 56.511        | 118.72     | 174.88         | 136.81         | 184.78         |
| 0.8               | 64.584        | 123.85     | 188.17         | 142.94         | 198.10         |
| 0.9               | 72.657        | 129.51     | 202.04         | 149.57         | 212.07         |
| 1.0               | 80.730        | 135.59     | 216.40         | 156.58         | 226.56         |
Table 4. Natural frequencies of rotating toroidal shell, $\omega$ [Hz], $R = 1$ m, $a = 0.4$ m, $h = 0.01$ m, $n = 2$, $\omega_0 = 80.73$ Hz, $(N=20)$, engineering strains

<table>
<thead>
<tr>
<th>$\Omega/\omega_0$</th>
<th>$\Omega$ [Hz]</th>
<th>Asymmetric</th>
<th>Symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Forward $n = -2$</td>
<td>Backward $n = 2$</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>111.11</td>
<td>111.11</td>
</tr>
<tr>
<td>0.1</td>
<td>8.073</td>
<td>108.15</td>
<td>116.11</td>
</tr>
<tr>
<td>0.2</td>
<td>16.146</td>
<td>107.17</td>
<td>123.10</td>
</tr>
<tr>
<td>0.3</td>
<td>24.219</td>
<td>108.02</td>
<td>131.93</td>
</tr>
<tr>
<td>0.4</td>
<td>32.292</td>
<td>110.47</td>
<td>142.38</td>
</tr>
<tr>
<td>0.5</td>
<td>40.365</td>
<td>114.28</td>
<td>154.21</td>
</tr>
<tr>
<td>0.6</td>
<td>48.438</td>
<td>119.21</td>
<td>167.20</td>
</tr>
<tr>
<td>0.7</td>
<td>56.511</td>
<td>125.05</td>
<td>181.14</td>
</tr>
<tr>
<td>0.8</td>
<td>64.584</td>
<td>131.62</td>
<td>195.84</td>
</tr>
<tr>
<td>0.9</td>
<td>72.657</td>
<td>138.76</td>
<td>211.17</td>
</tr>
<tr>
<td>1.0</td>
<td>80.730</td>
<td>146.35</td>
<td>226.99</td>
</tr>
</tbody>
</table>

Table 5. Natural frequencies of thin-walled toroidal ring, $\omega$ [Hz], $R = 1$ m, $a = 0.05$ m, $h = 0.001$ m, $(N = 20)$

<table>
<thead>
<tr>
<th>Mode no.</th>
<th>Mode type</th>
<th>$n$</th>
<th>RRM</th>
<th>FEM 50×124 FE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>In-plane</td>
<td>2</td>
<td>39.229</td>
<td>39.276</td>
</tr>
<tr>
<td>2</td>
<td>Out-of-plane</td>
<td>2</td>
<td>42.008</td>
<td>41.922</td>
</tr>
<tr>
<td>3</td>
<td>In-plane</td>
<td>3</td>
<td>108.378</td>
<td>108.810</td>
</tr>
<tr>
<td>4</td>
<td>Out-of-plane</td>
<td>3</td>
<td>114.416</td>
<td>111.770</td>
</tr>
</tbody>
</table>

Table 6. Ultimate inside pressure and rotation speed of toroidal shell, $R = 1$ m, $a = 0.4$ m, $h = 0.01$ m

<table>
<thead>
<tr>
<th>Item</th>
<th>Material Steel</th>
<th>Yielding stress, $R_e$ (N/mm$^2$)</th>
<th>Ultimate pressure $p_u$ (MPa)</th>
<th>Ultimate rotation speed $\Omega_u$, Hz</th>
<th>$\Omega_u/\omega_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S 235</td>
<td>235</td>
<td>4.41</td>
<td>19.67</td>
<td>0.244</td>
</tr>
<tr>
<td>2</td>
<td>TSTE 390</td>
<td>390</td>
<td>7.31</td>
<td>27.66</td>
<td>0.343</td>
</tr>
<tr>
<td>3</td>
<td>TSTE 690V</td>
<td>690</td>
<td>12.94</td>
<td>48.95</td>
<td>0.606</td>
</tr>
</tbody>
</table>
7. List of figures

Fig. 1. Moving coordinate system of a rotating shell of revolution

Fig. 2. The rotating toroidal shell, main dimensions and displacements

Fig. 3. Membrane forces due to internal pressure and centrifugal load

Fig. 4. The first six mode shapes of toroidal shell cross-section, – - - U, - - - V, —— W

Fig. 5. The first six natural modes of toroidal shell (ABAQUS)

Fig. 6. Natural modes of FEM model in the coordinate planes (ABAQUS)

Fig. 7. Natural mode 4, \( n = 3 \), a) in-plane cross-sectional displacement:

--- RRM, \( \delta = U \hat{e}_i + W \hat{e}_j \); - - - FEM, \( \tilde{\delta} = \delta_i \hat{i} + \delta_j \hat{k} \); b) deformation in \( y-z \) plane,

--- RRM, - - - FEM

Fig. 8. Natural frequencies of the rotating toroidal shell, asymmetric modes,

--- RRM-ST, – - RRM-MT, - o - FEM, \( \Leftarrow \) eng. strains

Fig. 9. Natural frequencies of the rotating toroidal shell, symmetric modes,

--- RRM-ST, – - RRM-MT, - o - FEM, \( \Leftarrow \) eng. strains

Fig. 10. Shell cross-section deformation due to centrifugal load, \( \Omega = 60 \text{ rad/s} \),

a) RRM, \( \tilde{\delta} = U \hat{e}_i + W \hat{e}_j \); b) FEM, \( \bar{PP}' = \delta_i \hat{i} + \delta_j \hat{k} \).

Fig. 11. Tension forces of rotating toroidal shell, \( \Omega = 60 \text{ rad/s} \),

--- shell theory, – - membrane theory

Fig. 12. Natural modes of thin-walled toroidal ring (ABAQUS)

Fig. 13. Natural modes of thin-walled toroidal ring in orthogonal planes (ABAQUS)

Fig. 14. Natural frequencies of rotating thin-walled toroidal ring, in-plane modes,

--- RRM-ST, – - RRM-MT, - o - FEM, \( \Leftarrow \) eng. strains

Fig. 15. Natural frequencies of rotating thin-walled toroidal ring, out-of-plane modes,

--- RRM-ST, – - RRM-MT, - o - FEM, \( \Leftarrow \) eng. strains

Fig. 16. Toroidal shell cross-section displacements of antisymmetric mode, \( n = 2, \ \omega = 110.63 \text{ Hz} \)

Fig. 17. Toroidal shell cross-section displacements of symmetric mode, \( n = 2, \ \omega = 122.58 \text{ Hz} \)

Fig. 18. Toroidal shell, Fourier coefficients; a) antisymmetric mode, \( n = 2, \ \omega = 110.63 \text{ Hz} \),

b) symmetric mode, \( n = 2, \ \omega = 122.58 \text{ Hz} \)
Fig. 19. Natural frequencies of toroidal shell, symmetric mode 3, $n = 2$, Figs. 5 and 6: —— linear, RRM; - - - linear, FEM (CATIA); – · – nonlinear, FEM (ABAQUS); o , $R_e$ limit (Table 6), $\Leftarrow$ eng. strains
Figures
Figure 2
Figure 4
Figure 5
Figure 6

$\omega_1 = 80.87 \text{ Hz, } n = 0$

$\omega_2 = 110.63 \text{ Hz, } n = 2$

$\omega_3 = 122.58 \text{ Hz, } n = 2$

$\omega_4 = 207.28 \text{ Hz, } n = 3$

$\omega_5 = 207.75 \text{ Hz, } n = 3$

$\omega_6 = 311.75 \text{ Hz, } n = 4$
Figure 7
Figure 8
Figure 9
Figure 10
Figure 11
Figure 12

\[ \omega_1 = 39.276 \text{ Hz, } n = 2 \]

\[ \omega_2 = 41.922 \text{ Hz, } n = 2 \]

\[ \omega_3 = 108.810 \text{ Hz, } n = 3 \]

\[ \omega_4 = 111.770 \text{ Hz, } n = 3 \]
Figure 13
Figure 14
Figure 15
Figure 16
Figure 17
Figure 18
Figure 19