Nonlinear Euler–Bernoulli beam kinematics in progressive collapse analysis based on the Smith's approach

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ABSTRACT

Within the scope of this article a nonlinear kinematics of the two-dimensional, non-shear-deformable and extensible Euler–Bernoulli beam imposed with the planar flexure and/or lengthening/shortening is considered. The complete and exact formulations of the pertinent kinematic response quantities (displacements, curvature and strain) are derived and discussed. Special emphasis is given to the case of the symmetric bending devoid of the external longitudinal force action, since it represents an appropriate idealization of the realistic load cases for the most of the ship and aircraft structures. The relationship between proposed and conventional formulations, i.e. those commonly accepted in the current structural engineering practice and employed by the current progressive collapse analysis methods based on Smith’s approach, is discussed throughout the article and illustratively exemplified through the case of the pure symmetric bending of the Euler–Bernoulli cantilever. Finally, implications of the derived formulations pertinent to the progressive collapse analysis methods based on the Smith’s approach are discussed.

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1. Introduction

Since the inherent structural capability to resist bending loads is of the predominant importance for the ultimate limit state design and analysis of various monotonous thin-walled structures (e.g. ships,
airplanes, bridges, etc.), the ultimate load capacity expressed in terms of the ultimate bending moment can be considered as one of the most important structural safety measures in their conceptual synthesis. Although a very accurate and reliable ultimate load capacity assessment can be performed by utilization of the materially and geometrically nonlinear finite element analysis (further in text: NLFEA) of the entire (discretized) structural model, a significant effort and resources are required for an applicable realization of the procedure which would enable multiple automatic executions of the complete NLFEA sequence (pre-processing, processing, post-processing) for the myriad of various feasible design variants considered by the contemporary optimization based concept design procedures. Consequently, demand for utilization of the appropriate (sufficiently fast, robust and accurate) alternative progressive collapse analysis (further in text: PCA) methods still persists. Within the field of shipbuilding, among the most widely utilized alternative methods are various incremental-iterative PCA methods based either on the original conception (e.g. Refs. [1–5]) or further simplifications (e.g. Refs. [6–8]) of the Smith’s approach [9].

Since the Smith’s approach employs geometrically linear Euler–Bernoulli beam bending theory for evaluation of the hull girder bending load capacity, applicability of methods based on this approach is generally limited to evaluation of the structural flexure characterized by the small magnitude of the kinematic response quantities (displacements, curvature and strains). Consequently, utilization of the geometrically linear Euler–Bernoulli beam bending theory within the context of the PCA can be rendered valid only if the kinematic response quantities of the considered structure (idealized as a monotonous thin-walled beam) are of the sufficiently small magnitude during the whole range of the considered load (curvature) increments. Hence, the PCA of the monotonous thin-walled structures whose ultimate limit state is characterized by the appearance of large displacements should have a geometrically nonlinear character, i.e. it should be based on the complete and exact relationship between strains and displacements derived without the introduction of assumptions resulting with the simplification and/or disregard for effects of the nonlinear terms in the geometric equations. Furthermore, warping effects due to the longitudinal distributions of the transverse bending loads are regularly neglected by the current PCA methods, since the transverse cross section corresponding to the position of the maximum vertical bending moment (along the ship hull girder) is regularly considered to be the critical one.

Considering the above given remarks, purpose of the present article is to derive the complete and exact formulations of the pertinent response quantities of the non-shear-deformable and geometrically nonlinear Euler–Bernoulli beam kinematics and to investigate consequent implications pertinent to the PCA methods based on the Smith’s approach.

Lagrangian approach is used throughout the present paper, i.e. used coordinate system is defined with respect to the undeformed state (reference configuration) of the considered structure, as indicated by Fig. 1.

Positive sign characterizes displacements in a positive directions of the coordinate axes, angles with a counter-clockwise orientation and longitudinal normal strain induced by the lengthening. It is assumed that the longitudinal axis \( x \) lies in the vertical symmetry plane \((x, z)\) of the considered beam and that \( x \)-axis is orthogonal with respect to the principal axes of inertia of the arbitrary transverse cross section of the beam. Furthermore, considered problem is treated as a two-dimensional and planar, i.e. it is assumed that the change of all considered response fields with respect to the \( y \)-axis is nonexistent.

2. Curvature of a planar deflection curve

Kirchhoff’s assumptions are of the fundamental importance for consideration of the kinematics employed by the Euler–Bernoulli beam bending theory. Assumptions that transverse cross sections remain straight and infinitely rigid in their own plane imply that transverse cross sections are not deformed (warped nor lengthened/shortened) during the imposed flexure, but only translated and/or rotated with respect to their centroid. Assumption that transverse cross sections remain perpendicular on the deflected centroidal axis (further in text: deflection curve) actually introduces equality between angles \( \varphi \) and \( \chi \), see Fig. 1.
Scalar quantity (for the two-dimensional, planar problem) which represents the differential change of the angle $\varphi$ with respect to the arc length $s$ of the deflection curve (measured from an arbitrary point of the curve) is referred to as the curvature $\kappa$ of the deflection curve:

$$\kappa = \frac{d\varphi}{ds}$$  \hspace{1cm} (1)

Based on the geometrical relationships illustrated by Fig. 1, a following relationship can be established:

$$\tan \varphi = \frac{\frac{\partial w_0}{\partial x} dx}{dx - \left( -\frac{\partial u_0}{\partial x} dx \right)} = \frac{\frac{\partial w_0}{\partial x}}{1 + \frac{\partial u_0}{\partial x}}$$  \hspace{1cm} (2)

where $u_0$ and $w_0$ represent longitudinal and transverse components of the centroidal displacement vector, respectively. Eq. (2) implies that the angle $\varphi$ (with vertex at point $T_1'$) can be explicitly expressed through the components of the displacement gradient tensor:
\[ \varphi = \arctan \left( \frac{\partial w_0}{\partial x} \cdot \frac{1}{1 + \frac{\partial u_0}{\partial x}} \right) \]  

(3)

Differentiation of Eq. (3) with respect to \( x \) yields:

\[ \frac{d \varphi}{dx} = \frac{1 + \frac{\partial u_0}{\partial x}}{\left( 1 + \left( \frac{\partial u_0}{\partial x} \right)^2 \right)^{\frac{3}{2}}} \left( \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial w_0}{\partial x} \frac{\partial^2 u_0}{\partial x^2} \right) \]  

(4)

Furthermore, since in the limit case \( ds \) converges to \( dr \) (see Fig. 1), \( ds \) can be expressed by:

\[ ds = \sqrt{\left[ dx - \left( - \frac{\partial u_0}{\partial x} \right) dx \right]^2 + \left( \frac{\partial w_0}{\partial x} dx \right)^2} = dx \sqrt{\left( 1 + \left( \frac{\partial u_0}{\partial x} \right)^2 \right) + \left( \frac{\partial w_0}{\partial x} \right)^2} \]

which can be written in the following form:

\[ \frac{ds}{dx} = \left( 1 + \left( \frac{\partial u_0}{\partial x} \right)^2 + \left( \frac{\partial w_0}{\partial x} \right)^2 \right)^{\frac{1}{2}} \]  

(5)

Finally, incorporation of Eqs. (4) and (5) into Eq. (1) results in a nonlinear expression for the (planar) curvature (at an arbitrary point) of the deflection curve:

\[ \kappa = \frac{d \varphi}{dx} = \frac{1 + \frac{\partial u_0}{\partial x}}{\left( 1 + \left( \frac{\partial u_0}{\partial x} \right)^2 + \left( \frac{\partial w_0}{\partial x} \right)^2 \right)^{\frac{1}{2}}} \left( \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial w_0}{\partial x} \frac{\partial^2 u_0}{\partial x^2} \right) \]  

(6)

Eq. (6) was originally derived in Ref. [10] and subsequently in Ref. [11] using an alternative (one-dimensional) approach. Derivation based on the proposed two-dimensional kinematic model is given here for the sake of its comprehensive scrutiny and validation and since some relations are used further in the article. Since derivation of the Eq. (6) did not include any simplifications of the considered problem (e.g. introduction of the assumption regarding the small intensity of the angle \( \varphi \) and omission of the squared first order derivatives), the resulting Eq. (6) represents the complete and exact formulation of the deflection curve curvature (for the considered two-dimensional case), which is completely independent of the magnitude of \( u_0(x) \) and \( w_0(x) \). Introduction of the mentioned simplifications would reduce Eq. (6) to its linearized form characteristic for the geometrically linear Euler–Bernoulli beam bending theory employed by various PCA methods based on the Smith’s approach:

\[ \kappa_L = \frac{d^2 w_0}{dx^2} \]  

(7)

3. Total displacements

For the complete description of displacements for an arbitrary material point of the considered transverse cross section of the Euler–Bernoulli beam, cross sectional centroid displacements \( u_0(x) \) and \( w_0(x) \) should be consolidated with displacements \( u_1(x, z) \) and \( w_1(x, z) \), see Fig. 1, i.e. displacements due to the rotation of an arbitrary material point of the considered cross section about the cross sectional centroid:
Fig. 1, a following formulation is valid:

\[
\begin{align*}
    u(x, z) &= u_0(x) + u_1(x, z) \\
    w(x, z) &= w_0(x) + w_1(x, z)
\end{align*}
\]  

(8)

Considering the Eq. (2) and geometrical relationships given by Fig. 1, it can be noted that for displacements of an arbitrary material point \(T(x, z)\) (of the reference configuration) a following relationships are valid:

\[
\begin{align*}
    \sin \varphi &= \frac{-u_1(x, z)}{z} = \frac{\partial w_0(x)}{\partial x} \Rightarrow u_1(x, z) = -z \frac{\partial w_0(x)}{\partial x} \\
    \cos \varphi &= \frac{z - [- w_1(x, z)]}{z} = 1 + \frac{\partial u_0(x)}{\partial x} \Rightarrow w_1(x, z) = z \frac{\partial u_0(x)}{\partial x}
\end{align*}
\]  

(9)

Introduction of the Eq. (9) into Eq. (8) implies that the total displacements \(u(x, z)\) and \(w(x, z)\) of an arbitrary material point of the considered cross section can be expressed in a form entirely dependent only on \(u_0(x)\) and \(w_0(x)\):

\[
\begin{align*}
    u(x, z) &= u_0(x) - z \frac{\partial w_0(x)}{\partial x} \\
    w(x, z) &= w_0(x) + z \frac{\partial u_0(x)}{\partial x}
\end{align*}
\]  

(10)

4. Longitudinal normal strain

Previously mentioned kinematic assumptions, see Sec. 2, generally result in the annihilation of eight components of the strain tensor \((\varepsilon_{xy}, \varepsilon_{yz}, \gamma_{yx}, \gamma_{xz}, \gamma_{yx} = \gamma_{xz})\), where longitudinal normal strain \(\varepsilon_x\) remains as a sole existing component. Since the \(\varepsilon_x\) of the beam fiber \(z = 0\) (which coincides with the \(x\)-axis in the reference configuration) actually represents a relative change of the line segment \(T_1T_2\), see Fig. 1, a following formulation is valid:

\[
\varepsilon_x(x, z = 0) = \frac{T_1T_2 - T_1T_2}{T_1T_2} = \frac{\sqrt{(dx + \frac{\partial u_0}{\partial x} dx)^2 + \left(\frac{\partial w_0}{\partial x} dx\right)^2} - dx}{dx}
\]  

(11)

After some algebraic manipulation, Eq. (11) can be written in a following form:

\[
\varepsilon_x(x, z = 0) = \sqrt{1 + 2 \frac{\partial u_0}{\partial x} + \left(\frac{\partial u_0}{\partial x}\right)^2 + \left(\frac{\partial w_0}{\partial x}\right)^2} - 1
\]  

(12)

By introduction of the following substitution:

\[
\varepsilon_{x0} = 2 \frac{\partial u_0}{\partial x} + \left(\frac{\partial u_0}{\partial x}\right)^2 + \left(\frac{\partial w_0}{\partial x}\right)^2
\]  

(13)

Eq. (12) can be written in a following form:

\[
\varepsilon_x(x, z = 0) = \sqrt{1 + \varepsilon_{x0} - 1}
\]  

(14)

Furthermore, minuend in the Eq. (14) can be expanded into the binomial series with a positive exponent (convergent for \(|\varepsilon_{x0}| \leq 1\). Consequently, Eq. (14) can be written in a following form:
\[
\varepsilon(x, z = 0) = \frac{1}{2} e_{x0} - \frac{1}{8} e_{x0}^2 + \frac{1}{16} e_{x0}^3 - \frac{5}{128} e_{x0}^4 + \frac{7}{256} e_{x0}^5 - \frac{21}{1024} e_{x0}^6 + \ldots (15)
\]

Introduction of the Eq. (13) into Eq. (15) yields:

\[
\varepsilon(x, z = 0) = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u_0}{\partial x} \right)^2 + \left( \frac{\partial w_0}{\partial x} \right)^2 \right]\]

\[
- \frac{1}{8} \left[ 2 \left( \frac{\partial u_0}{\partial x} \right)^2 + \left( \frac{\partial w_0}{\partial x} \right)^2 \right] + \frac{1}{16} \left[ 2 \left( \frac{\partial u_0}{\partial x} \right)^2 + \left( \frac{\partial w_0}{\partial x} \right)^2 \right]^3 - \frac{5}{128} \left[ 2 \left( \frac{\partial u_0}{\partial x} \right)^2 + \left( \frac{\partial w_0}{\partial x} \right)^2 \right]^4 + \ldots (16)
\]

It can be noted that the omission of all higher order terms of the series given by the Eq. (16) results in its reduction onto a diagonal component \( E_{x0x0} \) of the Green–Lagrange strain tensor \( E_0 = 1/2(u_0^2 + v_0^2 + u_0^2 - v_0^2) \) (for the deflection curve), where \( u_0 = [u_0, v_0, 0, 0] \) represents the displacement vector of the considered material point of the deflection curve. Hence, it can be concluded that the longitudinal component of the Green–Lagrange strain tensor does not represent the complete and exact formulation of the longitudinal normal strain for the fiber at the position of the deflection curve. The same conclusion is also valid for an arbitrary fiber, which can be deduced from the generalized form of the Eq. (16) and derived further in the article. Furthermore, disregard for the squared powered derivatives within the first term of the series yields linearized relationship between longitudinal normal strain and the longitudinal displacement, i.e. one of the six Cauchy’s equations, which is usually used in the analysis of problems characterized by the small magnitudes of the strain and/or displacement gradient components (order of magnitude 0.001).

Complete formulation of the longitudinal normal strain for an arbitrary fiber at the position of the considered transverse cross section of the Euler–Bernoulli beam imposed with the general case of the planar load can be derived considering the total displacements \( u(x, z) \) and \( w(x, z) \) of an arbitrary material point \( T(x, z) \) of the considered transverse cross section within the scope of the approach analogous to the one employed for derivation of the Eq. (12):

\[
\varepsilon(x, z) = \sqrt{1 + 2 \left( \frac{\partial u}{\partial x} + \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right) - 1} (17)
\]

Differentiation of the Eq. (10) with respect to \( x \) results in the following total displacement derivatives for the considered material point:

\[
\frac{\partial u(x, z)}{\partial x} = \frac{\partial u_0(x)}{\partial x} - z \frac{\partial^2 w_0(x)}{\partial x^2}
\]

\[
\frac{\partial w(x, z)}{\partial x} = \frac{\partial w_0(x)}{\partial x} + z \frac{\partial^2 u_0(x)}{\partial x^2}
\]

whose introduction into Eq. (17) yields:
\[ \varepsilon_{x}(x, z) = \left\{ 1 + 2 \frac{\partial u_{0}}{\partial x} + \left( \frac{\partial u_{0}}{\partial x} \right)^{2} + \frac{\partial w_{0}}{\partial x} \right\}^{2} + 2z \left( \frac{\partial u_{0}}{\partial x} \right)^{2} + \frac{\partial^{2} w_{0}}{\partial x^{2}} - \frac{\partial w_{0}}{\partial x} \frac{\partial^{2} u_{0}}{\partial x^{2}} \right\} + 2z^{2} \left( \frac{\partial^{2} u_{0}}{\partial x^{2}} \right)^{2} + \left( \frac{\partial^{2} w_{0}}{\partial x^{2}} \right)^{2} \right\}^{\frac{1}{2}} - 1 \] \tag{19}

Eq. (19) represents the complete and exact formulation of the \( \varepsilon_{x}(x, z) \) for an arbitrary fiber at the considered transverse cross section of the extensible Euler–Bernoulli beam submitted to the planar flexure and/or shortening/lengthening. It is obvious that ability to determine \( \varepsilon_{x}(x, z) \) depends on the knowledge of \( u_{0}(x) \) and \( w_{0}(x) \).

For the case when considered structure is imposed only with the external axial (longitudinal) force, it is obvious that only shortening/lengthening devoid of bending will occur, i.e. that \( w_{0}(x) \) will not exist for the arbitrary intensity of the imposed load. This is true since one of the fundamental assumptions of all PCA methods based on Smith’s approach is that considered monotonous thin-walled structure, i.e. the ship hull girder, is designed in such a manner that its ultimate limit state is imminently determined by the inter-frame collapse of the hull girder’s longitudinal structure. This effectively disables occurrence of the global hull girder buckling and/or any other more complex hull girder collapse mode which might encompass any larger structural portion (more than one longitudinal structural segment).

Hence, in this case, Eq. (19) is reduced to one of the previously mentioned Cauchy’s equations, due to the nonexistence of \( w_{0}(x) \):

\[ \varepsilon_{x}(x, z) = \sqrt{1 + 2 \frac{\partial u_{0}}{\partial x} + \left( \frac{\partial u_{0}}{\partial x} \right)^{2} + \frac{\partial w_{0}}{\partial x} - 1} = \sqrt{\left( \frac{\partial u_{0}}{\partial x} + 1 \right)^{2}} - 1 = \frac{\partial u_{0}}{\partial x} \] \tag{20}

Considering the context of the present paper, it is much more interesting to scrutinize the case when considered structure is imposed with the planar load devoid of any external axial (longitudinal) force, i.e. when the intensity of such a force and/or its effects is negligible with respect to the intensity of the imposed bending load and/or its effects (e.g. realistic loading cases for the ship hull girder and/or aircraft wing). In this case \( \varepsilon_{x}(x, z) \) for the fiber \( z = 0 \) is equal to null:

\[ \varepsilon_{x}(x, z = 0) = 0 \] \tag{21}

This fact enables direct derivation of the relationship between relevant components of the cross sectional centroid displacement gradients. Hence, introduction of the Eq. (21) into Eq. (12) and its subsequent introduction into Eq. (13) yields \( \varepsilon_{x0} = 0 \) or:

\[ \left( \frac{\partial u_{0}}{\partial x} \right)^{2} + 2 \frac{\partial u_{0}}{\partial x} + \left( \frac{\partial w_{0}}{\partial x} \right)^{2} = 0 \] \tag{22}

Eq. (22) can be perceived as a quadratic equation with the parameter \( \frac{\partial u_{0}}{\partial x} \) and a following pertinent root:

\[ \left( \frac{\partial u_{0}}{\partial x} \right) = -1 + \sqrt{1 - \left( \frac{\partial w_{0}}{\partial x} \right)^{2}} \] \tag{23}

Furthermore, differentiation of the Eq. (23) with respect to \( x \) results in:

\[ \frac{\partial^{2} u_{0}}{\partial x^{2}} = \frac{\partial w_{0}}{\partial x} \frac{\partial^{2} w_{0}}{\partial x^{2}} \left[ 1 - \left( \frac{\partial w_{0}}{\partial x} \right)^{2} \right]^{\frac{1}{2}} \] \tag{24}
Introduction of Eqs. (23) and (24) into the Eq. (6) results in (after some rearrangement):

$$\kappa = \frac{\partial^2 w_0}{\partial x^2} \left[ 1 - \left( \frac{\partial w_0}{\partial x} \right)^2 \right] \frac{1}{2}$$

(25)

Eq. (25) actually represents the complete and exact formulation of the two-dimensional, planar curvature for the loading case devoid of the external axial (longitudinal) force action. It should be noted that even though $u_0(x)$ generally exists in this case, this formulation depends solely on the knowledge of $w_0(x)$. Furthermore, it should be noted here that an existing expression for curvature:

$$\kappa = \frac{\partial^2 w_0}{\partial x^2} \left[ 1 - \left( \frac{\partial w_0}{\partial x} \right)^2 \right] \frac{1}{2}$$

is based on the simplified Euler–Bernoulli beam bending kinematics, which omits occurrence of the longitudinal displacements during bending. Although kinematical model proposed in this article is not based on this assumption, this simplification can be accommodated, since Eq. (6) is reduced to this form in the case when $u_0(x) = 0$. Yet, when consistent consideration of the bending devoid of the external axial force is considered, as proposed in this article, expression given by Eq. (25) is obtained.

Introduction of Eqs. (22)–(25) into Eq. (19) results in a relationship of the following form:

$$\varepsilon_x(x, z) = \sqrt{(2\kappa)^2 - 2z\kappa + 1} = \sqrt{(-z\kappa + 1)^2 - 1} = -z\kappa$$

(26)

Eq. (26) represents the complete and exact formulation of the $\varepsilon_x(x, z)$ for an arbitrary fiber at the position of the considered transverse cross section of the Euler–Bernoulli beam for the general case of the symmetric bending devoid of the external axial (longitudinal) force action. It is important to notice that the expression $\varepsilon_{xL}(x, z) = -z\kappa_L$ used by virtually all of the PCA methods based on the Smith’s approach differs from the Eq. (26) only by the employed formulation of the curvature.

5. Example of application

For the case of the pure symmetric bending of the monotonous Euler Bernoulli beam, deflection curve assumes the form of the planar curve with the constant curvature, i.e. arc of the circle.

Fig. 2 illustrates geometrical relationships which enable direct determination of $w_0(x)$ and $u_0(x)$ for the case of the pure symmetric bending of the considered monotonous cantilever (e.g. rough beam idealization of the aircraft half wing):

$$OT_1 = OT_1' = x = R\chi$$

(27)

$$u_0(x) = R \sin \chi - x = R \sin \frac{x}{R} - x$$

$$w_0(x) = R - R \cos \chi = R \left( 1 - \cos \frac{x}{R} \right)$$

(28)

Expressions given by Eq. (28) define the deflection curve of the considered cantilever, while their first two derivations with respect to $x$ have a following form:

$$\frac{\partial u_0}{\partial x} = \cos \frac{x}{R} - 1; \quad \frac{\partial^2 u_0}{\partial x^2} = -\frac{1}{R} \sin \frac{x}{R}$$

$$\frac{\partial w_0}{\partial x} = \sin \frac{x}{R}; \quad \frac{\partial^2 w_0}{\partial x^2} = \frac{1}{R} \cos \frac{x}{R}$$

(29)
Introduction of the Eq. (29) into Eq. (6) reduces formulation of the deflection curve curvature to the following form:

\[ \kappa = \frac{1}{R} \]  

(30)

It should be noted that Eq. (30) represents formulation of the curvature which is virtually identical to the formulation used by all of the PCA methods based on the Smith’s approach. Furthermore, equality between \( \kappa \) and \( \kappa_L \) for the case of pure bending also induces equality between \( \varepsilon_x \) and \( \varepsilon_{xL} \), since a different curvature formulation employed by \( \varepsilon_x \) and \( \varepsilon_{xL} \) was the sole difference in their expressions of the otherwise identical form.

Considering the context of this article, it would be useful to scrutinize the relationship between the previously derived \( \varepsilon_x \) and a Green—Lagrange longitudinal normal strain which is commonly suggested (e.g. Refs. [12–13]) as a more accurate when dealing with the large displacements. For this purpose, expansion of the Eq. (17) into binomial series with a positive exponent results in:

\[
\varepsilon_x(x, z) = \left\{ \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right] \right\} - \\
\frac{1}{8} \left[ 2 \frac{\partial u}{\partial x} + \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right]^2 + \\
+ \frac{1}{16} \left[ 2 \frac{\partial u}{\partial x} + \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right]^3 - \\
- \frac{5}{128} \left[ 2 \frac{\partial u}{\partial x} + \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right]^4 + \ldots
\]

(31)

Furthermore, introduction of Eqs. (18), (29) and (30) into Eq. (31) yields:
\[ \varepsilon_x(x, z) = \frac{1}{2} [(zk)^2 - 2zk] - \frac{1}{8} [(zk)^2 - 2zk]^2 + \frac{1}{16} [(zk)^2 - 2zk]^3 - \ldots \] (32)

Since the first term of the series in Eqs. (31) and (32) represents the Green–Lagrange longitudinal normal strain, it can be concluded that the Green–Lagrange longitudinal normal strain represents only one portion of the complete formulation. In this respect, Fig. 3 illustrates distributions of the first six terms of the series given by the Eq. (32) for the arbitrary transverse cross section of the considered beam (with the unitary height \( H \) and length to height ratio \( L/H \)) imposed with the curvature intensity of 0.95 m\(^{-1}\), while Fig. 4 illustrates distributions of the total sum and the first six partial sums of the series. Extreme intensities of the mentioned beam parameters were intentionally used in order to resolutely display and emphasize the character of the displayed distributions.

It can be noted that the Green–Lagrange longitudinal normal strain is nonlinearly and asymmetrically (with respect to the longitudinal neutral line) distributed over the transverse cross section of the beam. Furthermore, it should be noted that its contribution to the total sum of the infinite, but convergent, series is the greatest among all terms of the series. However, the alternating sign of the successive terms of the series induces progressive linearization and symmetrization of the resulting distribution, as propagation towards the higher partial sums of the series takes place. Finally, by summation of all terms of the series, resulting distribution becomes completely linear and symmetric, as suggested by Eq. (26). Hence, it can be concluded that in contrast to the nonlinearly distributed Green–Lagrange longitudinal normal strain approximation, linearly distributed \( \varepsilon_{xL} \) represents the exact or completely accurate strain formulation for the considered large displacement case.

In contrast to the longitudinal normal strain, deflection curves obtained by the exact and conventional (linear) approach to the considered problem are not the same. Conventional approach omits longitudinal displacements, while transverse displacements are determined by a twofold integration of the linearized curvature. Thereby, value of the both integration constants (obtained in accordance with the boundary conditions: \( w_0(0) = 0, \partial w_0(0)/\partial x = 0 \) of the considered problem) is equal to null. Hence, transverse displacements are given by:

\[ w_{0L}(x) = \frac{\kappa_L x^2}{2} \]

Since within the framework of the conventional approach \( \kappa_L = 1/R \), conventionally determined transverse displacements can be written in a following form:

\[ w_{0L}(x) = \frac{1}{R} \frac{x^2}{2} = \frac{R}{2} \left( \frac{1}{R} x \right)^2 \] (33)

Alternatively, considered problem enables exact predetermination of the displacements according to Eq. (28) within the framework of the exact approach. For the sake of the comparison, cosine within the expression for \( w_0(x) \) can be expanded into the infinite series, resulting in the following equation:
It is obvious that Eq. (33) represents only the first term of the infinite series given by Eq. (34) which, in a turn, exactly describes transverse displacement of the beam's material points which coincide with its deflection curve. Hence, a logical question emerges: what is the extent of the difference among transverse displacements obtained by those two different approaches in the context of the considered problem? In that respect, Fig. 5 illustrates a relative difference (% = 100(w0L − w0)/w0) of the transverse displacements obtained by the exact and the conventional approach to consideration of the pure symmetric beam bending, for a curvature range from 0.0005 m to 0.05 m and a beam length to height ratio (L/H) from 1 to 50, with beam height considered as unitary.

6. Discussion and conclusions

Within the scope of this article a geometrically nonlinear kinematics of the two-dimensional extensible Euler–Bernoulli beam imposed with the general case of the planar load is considered and the complete and exact general forms of the pertinent kinematic response quantities’ formulations, see
Eqs. (6), (10) and (19), are derived and discussed. It should be noted that all derived formulations are characterized by the sole dependence on \( u_0(x) \) and/or \( w_0(x) \) and by the complete independence of the magnitude of \( u_0(x) \) and/or \( w_0(x) \). This implies that one-dimensional problem consideration is entirely sufficient for the exact and complete two-dimensional description of the considered problem, which is valid in either case of small or large displacements. Furthermore, it is shown that Green—Lagrange strain, as well as its further particularizations (e.g. von Karman strains, commonly suggested for the case of moderately large rotations and small strains) represent incomplete approximations of \( \epsilon_x(x, z) \).

Special attention is given to the particular case of the symmetric bending devoid of the external longitudinal force action, since it represents an appropriate idealization of the realistic load cases for the most of the ship and aircraft structures. For this case the complete and exact formulations of the two-dimensional, planar curvature, see Eq. (25), and the longitudinal normal strain, see Eq. (26), are derived, which are dependant solely on \( w_0(x) \). Furthermore, it is shown that in this case \( \epsilon_x(x, z) \) is linearly distributed over the height of the transverse cross section, irrespective of the magnitude of the magnitude of \( u_0(x) \) and/or \( w_0(x) \). This has significant impact on the PCA of the structures submitted to the flexure characterized by the occurrence of the large displacements and/or curvatures, since this implies that linearly proportional relationship between \( \epsilon_x \) and \( x \) is universally and completely correct (for the non-shear-deformable case), i.e. that current PCA methods based on the Smith’s approach can be applied universally in an unmodified form, even when a more flexible ship structures (e.g. hull girders of large container ships) are considered. In other words, work presented by this article suggests that Smith’s ingenious PCA methodology is actually completely valid even when employed beyond the limitations of the conventional Euler—Bernoulli bending theory kinematics (i.e. small hull girder transverse displacements, complete omission of any longitudinal displacements) implicitly incorporated into the Smith’s approach.

Furthermore, incorporation of the proposed kinematics into the usual procedure for derivation of the Euler—Bernoulli beam bending equation results in a following relationships (for the case of the symmetric bending devoid of the longitudinal force action):

\[
\frac{\partial^2}{\partial x^2} \left[ (E_Iy)_E \frac{\partial^2 w_0}{\partial x^2} \right] = \frac{\partial^2}{\partial x^2} \left[ (E_Iy)_E \kappa \right] = -q_x(x) \tag{35}
\]

\[
M_y(x) = (E_Iy)_E \frac{\partial^2 w_0}{\partial x^2} = (E_Iy)_E \kappa \tag{36}
\]

where \( M_y(x) \) and \( q_x(x) \) represent the longitudinal distributions of the bending moment and the distributed load, respectively, while \( (E_Iy)_E \) represents an instantaneous or effective flexural stiffness. Eq. (35) represents the nonlinear Euler—Bernoulli beam bending differential equation, while Eq. (36) represents formulation of the stress resultant upon which the Smith’s approach is based. Unfortunately, it is not possible to obtain analytical solution, i.e. \( w_0(x) \) and/or its derivations, by integration of Eq. (35) nor (36) for some particular boundary conditions. However, this is irrelevant within the context of the PCA methods based on Smith’s approach, since they are based on the determination of the moment of the internal longitudinal forces (at the position of the considered transverse cross section) for the predetermined range of the progressively magnified curvatures, according to the Eq. (36), which renders the explicit determination of \( w_0(x) \) as unnecessary.

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