Rational magnification of the plate elastic shear buckling strength

Stanislav Kitarovic*, Jerolim Andric, Karlo Piric

University of Zagreb, Faculty of Mechanical Engineering and Naval Architecture, Ivana Lucica 5, 10000 Zagreb, Croatia

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**Abstract**

Various feasible approaches to magnification of the plate elastic shear buckling strength are comprehensively investigated. Based on derived theoretical envelopes of the considered approaches, stiffening parallel to the longer plate edges is identified as the most effective approach to the considered problem. An accompanying analytical formulation, convenient for utilization in structural analysis and design of the plated structures, is approximated. Various recommendations and formulations relevant for determination of the rational stiffened plate design (with respect to the considered problem) are proposed. All derived conclusions and proposed formulations are based on results of the numerous rationally designed numerical simulations.

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**1. Introduction**

During the ship exploitation, existing hull girder vertical shear loading is predominantly resisted by the vertically oriented parts of the hull girder structure (e.g. sides and longitudinal bulkheads). Generally, pronounced shear loading of the significant intensity can induce occurrence of the shear buckling and ultimately lead to the shear collapse of the structural components if acting edge shear load exceeds their ultimate shear load capacity. Possibility of incidence for such a hazardous event becomes even more significant if considered structural elements have been damaged previously (e.g. by collision) and consequently suffered a notable load capacity decrease with respect to the undamaged condition. In this context, consideration of the hull girder structural element’s shear load capacity can represent a relevant aspect and an important structural adequacy criterion in analysis and design of the ship structures. The same reasoning is also valid for other fields of the structural engineering dealing with the thin-walled structures whose components could be imposed with the edge shear load during the exploitation.

Occurrence of the shear buckling does not represent the ultimate limit state of the plate loaded by edge shear. For the case of thicker plates, local yielding of the compressed material always precedes occurrence of buckling. Conversely, for the case of thinner plates, a certain post-critical margin of the shear load capacity always exists after occurrence of the elastic buckling due to the diagonal tension field action arising along the tensile diagonals. Redistribution of the imposed loading induces increase of the tensile stress within the tensile diagonal fields, while compressive stresses remain unchanged within the compressive diagonal fields. Ultimate limit state of the plate loaded by edge shear is reached by a total depletion of the complete load capacity margin of the diagonal tension fields, i.e. by the complete plastification (yielding) of the plate material loaded in tension.

Unfortunately, the exact assessment of the plate ultimate shear strength necessitates detailed description of the still unresolved and very complex interaction of all relevant parameters of influence. This disables complete and accurate theoretical description of the shear collapse phenomena. Consequently, virtually all of the existing ultimate shear strength formulations are of the (semi)empirical nature and are predominantly formulated by means of the regression analyses based on various results obtained by experiments and/or numerical simulations. Various existing formulations of the plate ultimate shear strength can be found in [1–6].

Since the most of the contemporary ultimate shear strength formulations are based on the correction of the elastic shear buckling strength, various aspects and implications of the feasible approaches to the elastic shear buckling inhibition are discussed in this article. Perhaps the most obvious one among them is maximization of the rotational restraint along the plate edges, resulting ultimately with the clamped edge restraints. However, the actual edge restraint imposed at the plate edge joints of the realistic thin-walled structures varies between the two extreme cases (simply supported and clamped). Only the worst (simply supported) case, which provides the highest margin for the increase of the plate elastic shear strength, is considered in this article.

**1.1. Elastic shear buckling of the simply supported, isotropic, flat plate**

An isotropic, flat plate of length \( L \), breadth \( B \) and thickness \( t \), loaded along all boundaries (edges) by pure and uniformly distributed in-plane edge shear stress is considered (see Fig. 1).
If the geometrical and material characteristics of the considered plate are such that elastic shear buckling occurs at the critical intensity of the imposed edge shear loading, the corresponding deflected state of the unstable equilibrium can be described by the Saint Venant’s equation [7] reduced to the pure edge shear case, which is valid if the lateral displacements are small with respect to t:

\[
\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -2 \frac{\tau_{cr}^2}{D} \frac{\partial^2 w}{\partial x \partial y} \tag{1}
\]

where \( \tau_{cr} \) represents the plate elastic shear buckling stress, \( D = Et^3/(12(1-\nu^2)) \) denotes the plate flexural rigidity, \( w \) denotes the out-of-plane or lateral (in direction of the z-axis) displacements. Although the exact solution of Eq. (1) is not known, an approximate solution (which suggests appearance of the buckling half waves along the plate’s compressive diagonals) can be derived using the stationary potential energy principle and a Ritz method (lateral displacements expressed in terms of the appropriate coordinate functions which satisfy considered boundary conditions) [8]:

\[
\tau_{cr}^p = \frac{k_p \pi^2 D}{B^2 t} = \frac{k_p \pi^2 E}{12(1-\nu^2)} \left( \frac{t}{B} \right)^2 \tag{2}
\]

Form of the buckling half waves generally depends on the tensile forces acting along the tensile diagonals, while their number is proportional to the plate aspect ratio \( L/B \): \( E \) and \( \nu \) denote the Young’s modulus of elasticity and Poisson’s ratio, respectively, while \( k_p \) denotes the non-dimensional factor dependant only on \( L/B \):

\[
k_p = \begin{cases} 
5.34 + 4L/B - 2 & \text{for } L/B \geq 1 \\
5.34(L/B)^2 + 4 & \text{for } L/B < 1 
\end{cases} \tag{3}
\]

Although the formulation of \( k_p \) given by Eq. (3) is widely accepted in the contemporary structural analysis and design practice, it actually represents the parabolic approximation of a more accurate values of \( k_p \) calculated previously by various authors [9].

1.2. Magnification of the plate elastic shear buckling strength

It should be noted that according to Eq. (2) \( \tau_{cr}^p \) depends solely on the plate geometrical characteristics \( L/B \) and \( t/B \) for a given plate material. This suggests that \( \tau_{cr}^p \) can be increased by two distinct approaches: by decrease in the \( L/B \) value and/or by increase in the \( t/B \) value. Decrease in the \( L/B \) value can be accomplished by subdivision of the plate onto a smaller and equal parts by addition of the equidistantly placed transverse (parallel with \( x \)-axis) stiffeners, whereby \( t/B \) ratio remains unchanged. On the other hand, increase in the \( t/B \) value can be accomplished by the increase of the plate thickness and/or by subdivision of the plate onto a smaller and equal parts by addition of the equidistantly placed longitudinal (parallel with \( y \)-axis) stiffeners.

Derived formulations of the theoretical envelopes (along with their corresponding domains, valid for all feasible stiffened plate layouts) characteristic for the stiffening approach to the considered problem are given in Table 1 and illustrated (for the considered cases of stiffening by one, two and three stiffeners of the infinite flexural rigidity) by Fig. 2 (where thick curves represent valid envelopes, while thin curves represent plots of the equations given in Table 1, irrespective of the correspondingly valid domain). It is important to note and to emphasize that the proposed theoretical envelopes are qualitatively and quantitatively valid for an arbitrary plate thickness, i.e. \( t/B \) ratio.

**Fig. 1.** Elastic shear buckling of the simply supported (un)stiffened plate.
that an alternative stiffening approach is much more effective. Hence, in addition to the identification of the most effective approach to the magnification of the elastic shear buckling strength of the simply supported plate, the overall aim of this article is to provide a corresponding formulation valid for this approach to the considered problem in a sufficiently general and accurate manner.

2. Modeling of the considered problem

In order to validate the above given theoretical considerations and to provide the basis for the formulation proposed in Section 4, an overall total of 276 rationally designed numerical experiments are performed employing the finite element method (FEM) numerical simulations for generation of the results for various (un)stiffened plate configurations. All considered variants are characterized by the same (isotropic and linear-elastic) material \((E = 206 \text{ GPa}; \nu = 0.3)\) and \(B = 3200 \text{ mm}\). Furthermore, due to the previously mentioned symmetry of the considered problem, only variants characterized by \(L/B \geq 1\) are considered. Although all other geometrical properties fall within the (relatively wide) range determined to cover the characteristic dimensions of the ship hull girder side plating (flat single side plating bounded by the wing/ hopper tanks and web frames of the existing and variously sized single hull bulk carriers), obtained results are fully applicable in other fields of the structural engineering dealing with the thin-walled structures whose (flat) components could be imposed with the edge shear load during the exploitation.

Unstiffened plates are analyzed in order to investigate the correspondence of the results obtained by utilization of Eq. (2) and by numerical simulations. For this purpose five different \(L/B\) values \((1; 2; 3; 6; 12)\) and five different \(t/B\) values \((0.0025; 0.00375; 0.005; 0.00625; 0.0075)\) are considered, resulting in a total of 54 numerical experiments.

Furthermore, in order to verify the theoretically obtained envelopes (see Table 1 and Fig. 2) and their independence of \(t\), analyses of the plates stiffened by one, two or three longitudinal or transverse stiffeners (of the infinite flexural rigidity) are performed for three different \(L/B\) values \((1; 2; 3)\) and three different \(t/B\) values \((0.0025; 0.005; 0.0075)\), resulting in a total of 54 numerical experiments.

Finally, in order to generate the experimental basis for the proposed formulation, analyses of the plates stiffened by longitudinal stiffeners (of the finite flexural rigidity) are performed. In this respect, previously mentioned Timoshenko’s paradigm (which disregards stiffener’s torsional rigidity) enables arbitrary selection of the shape and scantlings of the employed stiffener profile, since only the stiffener’s moment of inertia \(I_\ell\) is relevant for its flexural rigidity. Among an infinite number of the possible stiffener profile variants characterized by the same particular \(I_\ell\), a flangeless (flatbar) stiffener profile is selected for further consideration. This choice is motivated by the irrelevance of the higher section modulus (offered by the flanged profiles) for the considered problem and since the flatbar stiffener offers the highest possible \(I_\ell\) for the particular stiffener cross sectional area \(A_\ell\), or in another words, provides attainment of the particular flexural rigidity with the least amount of the stiffener material. Furthermore, since the results \(\left(\frac{t^3}{\psi}B^2\right)\) are suggestively independent of \(t\), the same value \((16 \text{ mm})\) is used for the stiffened web thickness \(h_w\) and \(t\) of all variants considered within this experimental batch and various stiffener flexural rigidities considered are attained only by variation of the stiffener web height \(h_w\). Thereby, 13 different stiffener flexural rigidities are considered for each of three different \(L/B\) values \((1; 2; 3)\) and three different \(N_1\) values \((1, 2, 3)\), resulting in a total of 117 numerical experiments.

### Table 1

Theoretical envelopes characteristic for the stiffening approach to the considered problem.

<table>
<thead>
<tr>
<th>Stiffened plate layout</th>
<th>Longitudinal stiffening</th>
<th>Transverse stiffening</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L/B \leq 1) (a/b \leq 1)</td>
<td>(\frac{\sigma}{\sigma_{cr}} = \frac{5.34(1+4t/B)N_1/N_s^2}{5.34(1+4t/B)}) for (0 &lt; L/B \leq 1/(N_1 + 1))</td>
<td>(\frac{\sigma}{\sigma_{cr}} = \frac{5.34(N_1 + 1)^2 + 4t/B}{5.34(N_1 + 1)}) for (L/B \geq 1)</td>
</tr>
<tr>
<td>(L/B \leq 1) (a/b \geq 1)</td>
<td>(\frac{\sigma}{\sigma_{cr}} = \frac{5.34(1+4t/B)N_1/N_s^2}{5.34(1+4t/B)}) for (1/(N_1 + 1) \leq L/B \leq 1)</td>
<td>Not feasible</td>
</tr>
<tr>
<td>(L/B \geq 1) (a/b \leq 1)</td>
<td>Not feasible</td>
<td>(\frac{\sigma}{\sigma_{cr}} = \frac{5.34(N_1 + 1)^2 + 4t/B}{5.34(N_1 + 1)}) for (1 \leq L/B \leq N_1 + 1)</td>
</tr>
<tr>
<td>(L/B \geq 1) (a/b \geq 1)</td>
<td>(\frac{\sigma}{\sigma_{cr}} = \frac{5.34(1+4t/B)N_1/N_s^2}{5.34(1+4t/B)}) for (L/B \geq 1)</td>
<td>(\frac{\sigma}{\sigma_{cr}} = \frac{5.34(N_1 + 1)^2 + 4t/B}{5.34(N_1 + 1)}) for (L/B \geq N_1 + 1)</td>
</tr>
</tbody>
</table>

Generally, it can be noted that the transverse stiffening represents better approach for the \(L/B \leq 1\) cases, while longitudinal stiffening is preferable for the \(L/B \geq 1\) cases. Furthermore, it should be noted that inverse pairs of the \(L/B\) values (e.g. \(L/B = 2\) and \(L/B = 0.5\)) are characterized by the identical results. This is due to the inherent irrelevance of the acting edge shear load direction, which consequently induces overall symmetry of the obtained results with respect to the \(L/B = 1\) case, which is characterized by the lowest attainable critical shear stress increase. For this case identical results are obtained by longitudinal and transverse stiffening, since both of those approaches result in an identical stiffened plate layout. Hence, based on the above considerations, it can be generally concluded that the plate stiffening with stiffeners parallel to the longer side of the initial (i.e. unstiffened) plate always represents a more gainful approach, irrespective whether \(L/B < 1\) or \(L/B > 1\).
Throughout the article, a particular variant can be easily identified according to the assigned designation (e.g. AR1-T16-S1L designates the plate with $L/B = 1$, $t = t_w = 16$ mm and one longitudinal stiffener).

### 2.1. Numerical simulations

All executed numerical simulations are based on utilization of the FEM analysis of the discretized models of the considered (un)stiffened plate variants, whose bifurcation buckling load is determined by the eigenvalue analysis, as implemented within the employed FEMAP/NX Nastran [16] computer program. Lanczos method is used for the eigenvalue extraction and the lowest (positive) eigenvalue is accepted as the quantitative representative of the relevant buckling mode.

### 2.2. Discretized models

All considered models are discretized by the two-dimensional, quadrilateral, isoparametric finite elements with four nodes (CQUAD4) and six degrees of freedom (DoF) at each node. The imposed edge shear load is applied by means of the properly oriented nodal forces along the plate edges, as illustrated by Fig. 1. Their magnitude is determined according to the expressions given in Fig. 1, where $P_L$ and $P_B$ denote the absolute values of the nodal forces applied along the $L$ and $B$ edges, respectively. $F_{Corner}^L$ and $F_{Corner}^B$ denote the absolute values of the components (parallel to the $L$ and $B$ edges, respectively) of the resultant nodal force applied at the plate corners (nodes coincident with the points $P_1$, $P_2$, $P_3$ and $P_4$), while $A_L$ and $A_B$ denote the cross sectional areas of the $L$ and $B$ edges, respectively. $P^L$ and $P^B$ denote the absolute values of the total forces applied along the $L$ and $B$ edges, respectively, while $n^L$ and $n^B$ denote the total number of nodes along the $L$ and $B$ edges, respectively.

Indicated expressions for $F_{Corner}^L$, $P_L$, $F_{Corner}^B$, and $P_B$ are derived in order to ensure the proper in-plane deformation of the corner finite elements, i.e. to ensure that the straightness of the plate edges is fully retained in the deformed state of the (un)stiffened plate. In this respect, it can be noted that $P_L/P_{Corner}^L$ and/or $P_B/P_{Corner}^B$ ratios should be equal to 4.

In order to properly simulate deformation of the simply supported (un)stiffened plate models, nodal DoF constraints are applied as described by Table 2, where 0 and 1 denote constrained and unconstrained nodal DoF, respectively. For the numerical simulations of the simply supported plates reinforced by the stiffeners of the infinite flexural rigidity, in addition to the constraints given by Table 2, all nodes along the plate to stiffener joint (line segment $P_6P_7$ in Fig. 1) are characterized by the constrained vertical nodal displacement ($T_z = 0$) in order to properly simulate influence of the (unmodeled) stiffeners.

In order to rationally determine an appropriate finite element mesh density for discretization of all considered FEM models, an extensive mesh convergence study is performed. For this purpose, two extreme values of $L/B$ (1; 12) and $t/B$ (0.0025; 0.0075) are considered for both unstiffened and stiffened (by three stiffeners of the infinite flexural rigidity) plate variants, since the sensitivity of the results obtained for those cases should envelop the sensitivities characteristic for all other (un)stiffened plate variants considered by the previously given plan of numerical experiments. For each of those eight (un)stiffened plate variants, ten different mesh density variants are considered (with 12, 24, 36, 48, 60, 72, 84, 96, 108 and 120 elements along the $B$ edges). Corresponding number of elements along the $L$ edges is unambiguously determined so as to keep the finite element aspect ratio equal to 1 for each mesh density variant. This results in a total of 80 additional numerical experiments considered within the framework of the mesh convergence study.

![Fig. 2. Plots of the theoretical envelopes given by Table 1 for plates stiffened by one, two and three stiffeners (of the infinite flexural rigidity).](image-url)
Table 2
Constraints of the nodal DoFs.

<table>
<thead>
<tr>
<th>Node location (see Fig. 1)</th>
<th>Nodal DoFs:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T_x$ $T_y$ $T_z$ $R_x$ $R_y$ $R_z$</td>
</tr>
<tr>
<td>P$1$, P$2$, P$3$, P$4$, P$5$</td>
<td>1 1 0 1 1 0</td>
</tr>
<tr>
<td>P$6$, P$7$, P$8$, P$9$, P$10$</td>
<td>0 0 1 1 1 0</td>
</tr>
<tr>
<td>All other plate nodes</td>
<td>1 1 1 1 1 0</td>
</tr>
</tbody>
</table>

The results of the mesh convergence study are concisely given by Fig. 3. It can be noted that the mesh resolution characterized by 96 elements along the $B$ edges gives a considerable relative reduction ($>35\%$) in the total number of DoFs $N_{\text{DoF}}$ while always providing a very small relative difference ($<0.15\%$) of the obtained results ($\tau_{\text{cr}}$), all with respect to the finest mesh resolution. Consequently, this mesh resolution is used for all subsequently performed numerical simulations.

3. Results

Fig. 4 illustrates the results obtained for the considered unstiffened plates. An excellent agreement of the results obtained by Eq. (2) and numerical simulations can be observed. Although an approximated relationship between those results is given (see Fig. 4), only a negligible error is introduced by its disregard ($\tau^{\text{cr-ANALYTICAL}}_{\text{FEA}} = \tau^{\text{cr-ANALYTICAL}}_{\text{P}}$).

The results of the numerical simulations obtained for plates stiffened by the infinitely rigid stiffeners are given in Fig. 2. It should be observed that the results obtained for various $t/B$ ratios are practically coincident. Furthermore, a very good agreement with the proposed theoretical envelopes can be observed.

Superimposed display of all results obtained by the numerical simulations performed for all considered plate variants reinforced by stiffeners with various (finite) flexural rigidity is given in $\tau^{\text{cr}}_{\text{SP}}/\tau^{\text{cr}}_{\text{P}} - A_{\text{SP}}/A_{\text{P}}$ space, represented by Fig. 5, where $A_{\text{SP}}$ denotes the area of the transverse cross section of the longitudinally stiffened plate, while $A_{\text{P}}$ denotes the area of the transverse cross section of the initial (unstiffened) plate.

Fig. 5 also displays the results obtained by the plate thickness increase approach in $\tau^{\text{cr}}_{\text{SP}}/\tau^{\text{cr}}_{\text{P}} - A_{\text{SP}}/A_{\text{P}}$ space, where $\tau^{\text{cr}}_{\text{P}}$ denotes the elastic shear buckling strength of the plate characterized by the (increased) equivalent thickness $t + \bar{t}$ (see Fig. 5), while $A_{\text{P}}$ denotes the area of its transverse cross section.

A more detailed display of the results obtained for all considered stiffened plate variants is given in $\tau^{\text{cr}}_{\text{SP}}/\tau^{\text{cr}}_{\text{P}} - A_{\text{SP}}/A_{\text{P}}$ space, represented by Figs. 6–8, where $\bar{t}$ denotes the moment of inertia of the plating between stiffeners. Fig. 6 additionally indicates the results obtained by three existing formulations of the considered problem (i.e. Bleich [9], Hughes [14] and Alinia [15] formulations), which are applicable only for $L/B = 1$ cases among the considered ones.

4. Formulation of the elastic shear buckling strength of the longitudinally stiffened and simply supported flat plates made of isotropic material

Results obtained by the numerical simulations performed for nine considered plate variants reinforced by stiffeners of various (finite) flexural rigidity are used for formulation of the approximate description (surrogate model) of the considered problem. For this purpose a regression analysis based on utilization of the least squares method is performed, whereby coefficient of determination ($R^2$) is used as a quantitative measure of the accomplished approximation quality.

The crucial choice regarding the appropriate mathematical form of the approximation function is based on the similar ‘S-shaped’ scatter of the results obtained for every considered stiffened plate variant, which can be observed in Fig. 5. Although various sigmoid functions were considered for approximation of the results in $\tau^{\text{cr}}_{\text{SP}}/\tau^{\text{cr}}_{\text{P}} - A_{\text{SP}}/A_{\text{P}}$ space, the best approximate fit is accomplished by utilization of the adjusted cumulative distribution function of the Weibull distribution:

$$
\tau^{\text{cr}}_{\text{SP}}/\tau^{\text{cr}}_{\text{P}} = C_0 + \frac{1 - C_0}{\exp \left( \frac{A_{\text{SP}}/A_{\text{P}} - 1}{C_1} \right)^2} \quad C_1 \geq 0 \quad C_2 > 0
$$

where $C_0$, $C_1$, and $C_2$ represent the non-dimensional coefficients, whose specific values characterize every particular
stiffened plate variant considered. \( C_0 \) actually represents the highest \( \tau_{cr}^{SP}/\tau_{cr}^{P} \) value obtained (by numerical experiments) for each stiffened plate variant, while the corresponding values of \( C_1 \) and \( C_2 \) are determined numerically (using a Levenberg–Marquardt algorithm) within the framework of the least squares method. Since an unambiguous relationship between \( A^{SP}/A^P \) and \( t^P/t^P \) can be derived for the plates reinforced by the flatbar stiffeners:

\[
\frac{A^{SP}}{A^P} = 1 + N_S \left( \frac{l_w}{l} \right) \left( \frac{t^P}{t^P} \right)^2; \quad \frac{t^P}{t^P} = \left( \frac{A^{SP}}{A^P} - 1 \right) \frac{N_S + 1}{N_S} \left( \frac{B}{l_w} \right)^3
\]  

(6)

**Fig. 4.** Comparison of the unstiffened plate results obtained by FEM analysis and Eq. (2).

**Fig. 5.** Results obtained for the longitudinal plate stiffening and plate thickness increase.
Fig. 6. Results for AR1-T16: (a) S1L; (b) S2L; (c) S3L.

Fig. 7. Results for AR2-T16: (a) S1L; (b) S2L; (c) S3L.
where (i.e. capability) of the (un)stiffened considered problem. Equations (5) can be conveniently used for determination of the amount of additional material (introduced by stiffening parallel to the longer side of the plate) required for attainment of the required elastic shear buckling strength. However, since the solution of the particular structural design problem depends on the magnitude of the design load (i.e. demand) characteristic for the corresponding exploitational context of the considered structure, any propositions regarding the rational relationship between the stiffened plate design attributes ($\tau_{cr}^{SP}/\tau_{cr}$ and $A^{SP}/A$) relevant for the considered problem can be made.

Plots of Eq. (5) in $\tau_{cr}^{SP}/\tau_{cr} - A^{SP}/A$ space along with the accompanying values of $R^2$, while the corresponding plots of Eq. (7) in $\tau_{cr}^{SP}/\tau_{cr} - \tau_{cr}^{SP}$ space along with the calculated values of $C_0, C_1$ and $C_2$ are given in Figs. 6–8, for each of the nine considered stiffened plate variants.

In order to derive the general formulation valid for an arbitrary $L/B$ ratio within the considered range ($1 \leq L/B \leq 3$), all values of the $C_0, C_1$ and $C_2$ determined for the nine considered stiffened plate variants are used. Considering the number of the available results and their scatter (see Fig. 9), the second degree polynomial is selected as an adequate function for determination of the exact values of $C_i$ ($i = 0, ..., 2$) and $j = 1, ..., 3$

$$C_i = (k_0)_{i0} + (k_0)_{i1} \left( \frac{q}{B} \right) + (k_0)_{i2} \left( \frac{q}{B} \right)^2$$

(8)

where $(k_0)_{i0}$, $(k_0)_{i1}$ and $(k_0)_{i2}$ represent the (non-dimensional) second degree polynomial coefficients valid for the respective stiffened plate variant. Calculated values, which determine curves expressed in terms of $\tau_{cr}^{SP}/\tau_{cr}$ and $A^{SP}/A$ for each of the nine considered stiffened plate variants in $C_i - a/b$ space (illustrated by Fig. 9):

$$C_i = \left( \frac{q}{B} \right)_{i0} + \left( \frac{q}{B} \right)_{i1} \left( \frac{N_i}{B} \right) + \left( \frac{q}{B} \right)_{i2} \left( \frac{N_i}{B} \right)^2$$

(9)

It should be noted that the validity of the proposed formulation is limited on stiffening parallel to the longer side of the plate, with $N_i = 1, 2, 3$, while both $1 \leq L/B \leq 3$ and $0.333 \leq L/B \leq 1$ ranges are covered due to the previously mentioned symmetry of the considered problem (with respect to $L/B = 1$). However, within the $0.333 \leq L/B < 1$ range, $L$ and $a$ should be interchanged with $B$ and $b$, respectively.

5. Rational stiffened plate design with respect to the considered problem

Eq. (5) can be utilized to express the structural elastic shear buckling strength (i.e. capability) of the (un)stiffened flat structural components in a synthesis of various complex thin-walled structures. When a weight-critical structures are considered in this context, Eq. (5) can be conveniently used for determination of the amount of the additional material (introduced by stiffening parallel to the longer side of the plate) required for attainment of the required elastic shear buckling strength. However, since the solution of the particular structural design problem depends on the magnitude of the design load (i.e. demand) characteristic for the corresponding exploitational context of the considered structure, any propositions regarding the generally valid optimal stiffened plate design would be useless. Yet, some suggestions regarding the rational relationship between the stiffened plate design attributes ($\tau_{cr}^{SP}/\tau_{cr}$ and $A^{SP}/A$) relevant for the considered problem can be made.

Within the scope of the considered problem obvious design objectives are: maximization of $\tau_{cr}^{SP}/\tau_{cr}$ and minimization of $A^{SP}/A$. In this respect, a few interesting feasible designs represented by

Fig. 8. Results for AR3-T16: (a) S1L; (b) S2L; (c) S3L.
which the corresponding tangent line on the AP has the maximum value, i.e. where the
dervative: maximization of the ( \( \kappa \) ) curve. Consequently, this line must satisfy the following
mathematical form of the resulting equation:

\[
\left( \tau_{cr}^S / \tau_{cr}^P \right)_{DP2} = \frac{d(\tau_{cr}^S / \tau_{cr}^P)}{d(A^S/A^P)} \mid_{DP2} (A^S/A^P)_{DP2}
\]

within which Eq. (5) and its derivative can be introduced. Since the
the corresponding points on \( \tau_{cr}^S / \tau_{cr}^P (A^S/A^P) \) curve given by Eq. (5) could be identified. First among them, denoted as the Design point 1 (DP1), represents the inflection point of the \( \tau_{cr}^S / \tau_{cr}^P (A^S/A^P) \) curve, which is determined by:

\[
(A^S/A^P)_{DP1} = 1 + C_1 (1 - 1/C_2)^{1/C_2}
\]

DP1 is interesting since it represents the particular point at which the corresponding tangent line on the \( \tau_{cr}^S / \tau_{cr}^P (A^S/A^P) \) curve has the maximum value, i.e. where the \( \tau_{cr}^S / \tau_{cr}^P \) increase is maximal with respect to the \( A^S/A^P \) increase. Yet another interesting point, denoted as the Design point 2 (DP2), can be defined if the previously stated design objectives are condensed into one objective: maximization of the \( (\tau_{cr}^S / \tau_{cr}^P) / (A^S/A^P) \) ratio. In this case DP2 actually represents the particular point on the upper knee of the \( \tau_{cr}^S / \tau_{cr}^P (A^S/A^P) \) curve, in which the line that passes through the origin (of the \( \tau_{cr}^S / \tau_{cr}^P - A^S/A^P \) space) is tangential on the \( \tau_{cr}^S / \tau_{cr}^P (A^S/A^P) \) curve. Consequently, this line must satisfy the following equation:

\[
\left( \tau_{cr}^S / \tau_{cr}^P \right)_{DP2} = \frac{d(\tau_{cr}^S / \tau_{cr}^P)}{d(A^S/A^P)} \mid_{DP2} (A^S/A^P)_{DP2}
\]

Disables separation of the variable, i.e. since \( (A^S/A^P)_{DP2} \) cannot be expressed explicitly, Eq. (12) must be solved iteratively in order to determine \( (A^S/A^P)_{DP2} \).

Since Eq. (6) enables mapping of every design point from \( \tau_{cr}^S / \tau_{cr}^P - A^S/A^P \) into \( \tau_{cr}^S / \tau_{cr}^P - \rho_{cr}^P \) space, both DP1 and DP2, determined respectively by Eq. (10) and (12) for each of the nine previously considered stiffened plate variants, are indicated in Figs. 6–8, along with the results of the corresponding numerical simulations (18 additional numerical experiments).

Results given by Figs. 6–8 suggest that the feasible stiffened plate design represented by DP1 might be generally recommended as the lower bound of the preferred design set, even if the design load demands lesser structural capability, since the attainment of \( \tau_{cr}^S / \tau_{cr}^P \) requires relatively small increase in \( A^S/A^P \). In another words, with a small increase in the amount of the additional material (with respect to the amount required by the design load), stiffened plate's safety margin (regarding the edge shear load) can be increased substantially in all considered cases. Generally, all feasible designs characterized by \( A^S/A^P < (A^S/A^P)_{DP1} \) provide less

---

**Table 3**

<table>
<thead>
<tr>
<th>( N_0 = 1 ) (j = 1)</th>
<th>( N_0 = 2 ) (j = 2)</th>
<th>( N_0 = 3 ) (j = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_0)</td>
<td>(x_1)</td>
<td>(x_2)</td>
</tr>
<tr>
<td>C_0 (i = 0)</td>
<td>1.75756</td>
<td>0.66568</td>
</tr>
<tr>
<td>C_1 (i = 1)</td>
<td>-0.01043</td>
<td>0.03179</td>
</tr>
<tr>
<td>C_2 (i = 2)</td>
<td>2.13970</td>
<td>0.80864</td>
</tr>
</tbody>
</table>

---

**Fig. 9.** Coefficients C_0, C_1, and C_2.
then maximum $\tau_{cr}/\tau_{er}$ increase with respect to the $A_{cr}/A_{er}$ increase and a corresponding stiffener flexural rigidity is not sufficient to stabilize the plate in any significant manner (as indicated by Figs. 6–8), which consequently results with the mildly obstructed spread of the plate buckling half waves.

On the other hand, results given by Figs. 6–8 also suggest that the feasible stiffened plate design represented by DP2 might be recommended as the upper bound of the preferred design set. If the design load requires greater structural capability, a much more rational approach to this requirement is to increase the number of stiffeners. Generally, all feasible designs characterized by $A_{cr}/A_{er} > (A_{cr}/A_{er})_{DP2}$ provide increasingly smaller $\tau_{cr}/\tau_{er}$ increase with respect to the $A_{cr}/A_{er}$ increase. In another words, a substantial amount of the additional material is required to attain relatively small increase in the elastic shear buckling strength. It can be noted that in all considered cases stiffener flexural rigidity corresponding to DP2 is sufficient for substantial plate stabilization (as indicated by Figs. 6–8) and significant obstruction of the plate buckling half waves spread.

6. Conclusions

Within the scope of this article accuracy of the commonly and widely accepted formulation of the elastic shear buckling strength of the unstiffened flat plates, given by Eq. (2), is verified by the results obtained by numerical simulations (see Fig. 4).

Furthermore, proposed theoretical envelopes of the stiffening approach and their independence of $t$, derived using Timoshenko’s paradigm and Eqs. (2)–(4), are also verified by the obtained results of the numerical simulations (see Fig. 2). This confirms the proposition that longitudinal stiffening for $L/B \geq 1$ cases and transverse stiffening for $L/B \leq 1$ cases, i.e. stiffening parallel to the longer side of the plate, represents a more effective approach to the considered problem than the stiffening parallel to the shorter side of the plate. Moreover, comparison of the results obtained for all considered plates reinforced by stiffeners of various (finite) flexural rigidity with respect to the results obtained for the plate thickness increase approach (see Fig. 5), suggests that the proposed stiffening approach represents a more rational approach to the considered problem than the plate thickness increase approach. Hence, it can be unconditionally concluded that the proposed stiffening approach represents the most rational course for the plate elastic shear buckling strength magnification. Furthermore, based on the results of the rationally designed and properly configured numerical simulations, an approximate formulation for this stiffening approach is proposed (valid for $0.333 \leq L/B \leq 3$ and $N_s = 1, 2, 3$). In this respect, a considerably high accuracy level of the proposed formulation, as well as its convenient mathematical nature (relatively simple, analytical form), can contribute to its recognition as an useful additional tool in the process of analysis and/or design of the thin-walled structures.

Finally, proposed recommendations regarding the rational stiffened plate design (with respect to the considered problem) should not be perceived as definitive and exclusive since the discussed design points (DP1 and DP2) surely do not represent the only interesting design points. However, they can be considered as a reasonable guidelines for the rational decision making in design of the stiffened plates which could be imposed with the edge shear load during the exploitation.

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References